Enumeration of Contact Geometries for Part Registration in Design of Tactile Sensing Fixtures

Walter Willem Nederbragt and Bahram Ravani

Abstract— **This paper uses group theory for enumeration of contacts between geometric elements necessary for part registration and referencing in robotics. The results are applied to type synthesis and design of tactile sensing mechanical fixtures. Although the scope of the paper is limited to geometric contacts involving points, lines, planar surfaces, cylindrical surfaces, and spherical surfaces, the techniques developed are general and can be applied to other geometric features and non tactile sensing elements used in robotic calibration and part referencing.**

Keywords— **Enumeration, Group Theory, Contact Geometries, Fixturing, Part Registration, Part Referencing.**

I. Introduction

CONTACT sensing elements with mechanical fixtures are commonly used in robotics and manufacturing (see, for example, Duffie et al. [2], McCallion and Pham [7], and Slocum [14]) for part registration and referencing. This is the process of determining the relative location of a part with respect to another part or a world coordinate system. Part registration or referencing is also an essential step in robot calibration (see Roth, Mooring, and Ravani [13]).

In measuring relative locations between two bodies, mechanical fixtures are usually used to simplify the sensing functions and to improve repeatability. Much of the existing mechanical fixtures, for such a purpose, however, are one of a kind, ad hoc designs. There exists little work on enumeration of such fixtures. An exception to this is the work of McCallion and Pham [7], who used Kutzbach's criteria to aid them in the design of a few tactile sensing fixtures for robotic assembly. Other directly related work is that of Bicchi, Salisbury, and Brock [1] and Mason and Salisbury [6]. The former paper determines contact location from force measurments, and the latter defines contact types that can be used in design of robotic fingers. Both of these works provide excellent theoretical basis that can be used for certain design applications, especially when dealing with point contacts and design of tactile sensing robotic hands. These authors focused on the utilization of force information.

Recently Nederbragt and Ravani [9] have developed a more general design theory for design of tactile sensing mechanical fixtures. They exploit the symmetry of different measuring arrangements using group theory. The method is not limited to any specific type of contact or the use of force information. In this paper, we extend this work for type synthesis or enumeration of certain contact geometries that can be used for part registration. We then use these geometries as an aid in type synthesis and design of new mechaninical tactile sensing fixtures for part registration. Group therory has been used in the past in robotics (see, for example, Popplestone [11]) and in robotic assembly (see for example, Liu and Popplestone [5]). Much of the background can be found in Nederbragt and Ravani [9].

II. Part Referencing Based on Locations of Geometric Elements

Part registration or position referencing involves determining the kinematic relationship between two coordinate systems. This is important in robotics for automated part assembly (see, for example, McCallion and Pham [7]) and calibration (see, for example, Roth, Mooring, and Ravani [13]). In such applications the location of the end effector of a robot is measured by bringing it into contact with touch sensitive elements of a mechanical fixture or

W. W. Nederbragt is with the National Institute of Standards and Technology (NIST), Manufacturing Engineering Laboratory, Manufacturing System Integration Division, Gaithersburg, Maryland 20899
B. Ravani is with the Department of Mechanical and Aeronautical Engineering, University of California, Davis, California 95616

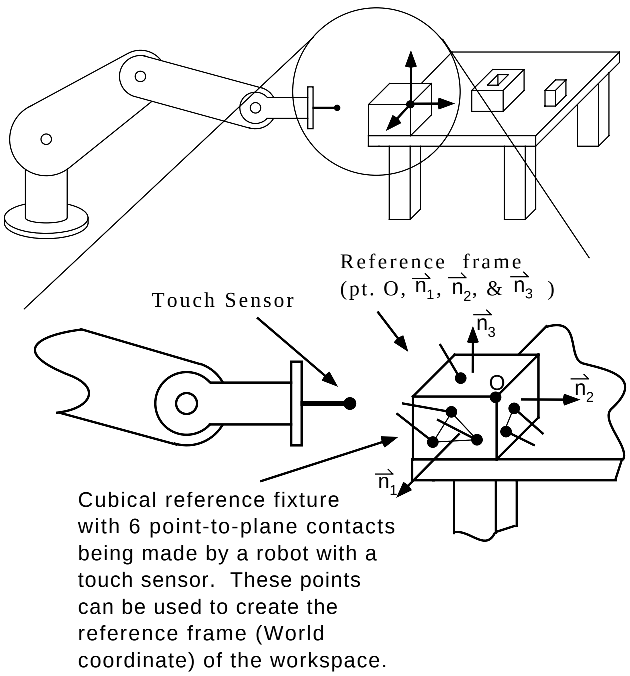

Fig. 1. The use of a cubical reference fixture

hand held touch sensitive elements in a robot are brought into contact with a mechanical fixture. Mathematically, the coincidence relationships between geometric elements between the two contacting elements are used to calculate the location of the end effector of the robot. Figure 1 illustrates this concept. In this figure, the relative location between a cube shaped fixture and a robot equipped with a touch sensor is being found using six point-to-planar surface contacts.

There are, in general, two categories of contacts. If the position of the contact is known in the local coordinate system of the geometric element, then we shall refer to the contact as a fixed contact (F). Otherwise, the contact is called a mobile contact (M). For example, if a point comes into contact with a planar surface, the contact is considered fixed if the location of the contact is known in the surface's frame and mobile otherwise. Note, a point is always a fixed contact because the location of a touch to its "body" must be the point itself. Using points, lines, cylindrical surfaces, spherical surfaces, and planar surfaces with both mobile and fixed contacts, a list of all possible contacts is shown in Table I.

TABLE I
Possible contacts between lines, spheres, planes, points, and cylinders

Objects in Contact	Type of Contact	Abbreviated Notation for Contact
Point - Point	F - F	P/F - P/F
Point - Sphere	F - F	P/F - S/F
Point - Sphere	F - M	P/F - S/M
Point - Plane	F - F	P/F - PL/F
Point - Plane	F - M	P/F - PL/M
Point - Line	F - F	P/F - L/F
Point - Line	F - M	P/F - L/M
Point - Cylinder	F - F	P/F - C/F
Point - Cylinder	F - M	P/F - C/M
Sphere - Sphere	F - F	S/F - S/F
Sphere - Sphere	F - M	S/F - S/M
Sphere - Sphere	M - M	S/M - S/M
Sphere - Plane	F - F	S/F - PL/F
Sphere - Plane	F - M	S/F - PL/M
Sphere - Plane	M - F	S/M - PL/F
Sphere - Plane	M - M	S/M - PL/M
Sphere - Line	F - F	S/F - L/F
Sphere - Line	F - M	S/F - L/M
Sphere - Line	M - F	S/M - L/F
Sphere - Line	M - M	S/M - L/M
Sphere - Cylinder	F - F	S/F - C/F
Sphere - Cylinder	F - M	S/F - C/M
Sphere - Cylinder	M - F	S/M - C/F
Sphere - Cylinder	M - M	S/M - C/M
Plane - Plane	F - F	PL/F - PL/F
Plane - Line	F - F	PL/F - L/F
Plane - Line	M - F	PL/M - L/F
Plane - Cylinder	F - F	PL/F - C/F
Plane - Cylinder	F - M	PL/F - C/M
Plane - Cylinder	M - F	PL/M - C/F
Plane - Cylinder	M - M	PL/M - C/M
Line - Line	F - F	L/F - L/F
Line - Line	F - M	L/F - L/M
Line - Line	M - M	L/M - L/M
Line - Cylinder	F - F	L/F - C/F
Line - Cylinder	F - M	L/F - C/M
Line - Cylinder	M - F	L/M - C/F
Line - Cylinder	M - M	L/M - C/M
Cylinder - Cylinder	F - F	C/F - C/F
Cylinder - Cylinder	F - M	C/F - C/M
Cylinder - Cylinder	M - M	C/M - C/M
KEY: F = fixed, M = mobile, P = point, L = line, S = sphere, PL = plane, C = cylinder		

In all of the cases studied here, it is assumed that the geometric elements are infinite (in other words, primitive elements). For example, a line would not have end points and a plane would not have edges. This means that if a line is in contact with a plane, it is lying in the plane, not intersecting it. This will be the case throughout this paper.

III. Group Theory Evaluation of Geometric Contacts

In this section, we develop a method for evaluation of contacts between the described geometric elements. The method used involves the use of the Euclidean group and its subgroups. Hence, each of the geometric element contacts is transformed into an equivalent group representa-

TABLE II
A PARTIAL LIST OF CONTINUOUS SUBGROUP CLASSES OF THE EUCLIDEAN GROUP

Notation	Description	D.O.F.
$\{T_u\}$	translation along line u	1
$\{R_u\}$	rotation about line u	1
$\{T_P\}$	translation on plane P	2
$\{C_u\}$	cylindrical motion about line u	2
$\{G_P\}$	planar motion on plane P	3
$\{S_o\}$	spherical motion about point o	3

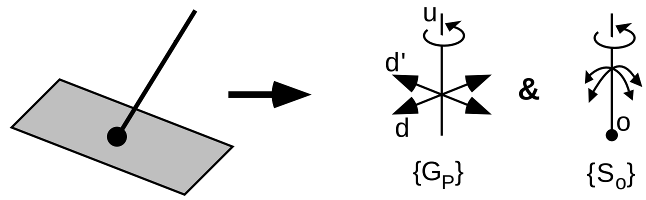

Fig. 2. A point to mobile planar surface contact

tion. Using group operations on the group representations, a method is constructed for testing combinations of these contacts for their usefulness in measuring the relative position between two bodies (or design of tactile sensing fixtures).

The process of finding the group representation for a contact between two geometric elements is a simple one. First the contact should be described in terms of allowable relative motions between contacting geometric elements. For example, rotation about a line (revolute motion), rotation about a point (spherical motion), translation along a line (prismatic motion), and rotation and translation along a line (cylindrical motion) represent four common contact motions. Then each of these motions can be described by their respective subgroups of the Euclidean group [3]. Table II lists continuous subgroup classes and notations being used in this paper. The resulting group representation for a contact is the composition of these subgroups.

In many instances, simplifications can be made to the group representation to make it more compact and more comprehensible. Given a compositions of two subgroups, it may be possible to join the two together and form a larger subgroup. For example, two linear translational groups $\{T_u\}$ and $\{T_{u'}\}$ can sometimes be joined to form one planar translational group, $\{T_u\}\{T_{u'}\} = \{T_P\}$. Moreover, some compositions of groups can be rewritten, resulting in a group of the same dimension. For example, two linear translational groups that are about the same line (line l) or parallel to line l can be composed into one linear translational group about the same line l. The group representation for a point - plane contact and a sphere - sphere contact are given below to further explain the derivation of the group representations.

Given a point - mobile plane contact (P/F-PL/M), the relative motion between the point and the plane can be described by two lower pair motions, a general planar motion and a spherical motion. The planar motion is due to the plane being mobile. Hence, it cannot sense the location of the touch, and, therefore, the point can be anywhere on the plane. The spherical motion comes from the point contact. Any rotations about the point have no influence on the contact condition. Therefore, spherical motions are possible. The group representation for this contact can be written as $\{G_P\}\{S_o\}$. However, a simplification can be made because there is a common subgroup shared between the two subgroups. The group $\{G_P\}$ can be represented by $\{T_d\}\{T_{d'}\}\{R_u\}$ where d and d' are intersecting, perpendicular lines and axis u is perpendicular to both d and d'. The subgroup $\{R_u\}$ is also contained in the subgroup $\{S_o\}$. Therefore, the group representation can be written as

$$(\{T_d\}\{T_{d'}\})(\{R_u\}\{S_o\}) = \{T_P\}\{S_o\}. \quad (1)$$

Figure 2 shows a picture of the contact and a schematic of the motion associated with that contact. The dimension of this group representation is five, two from $\{T_P\}$ and three from $\{S_o\}$. Note, if the planar surface was fixed that the resulting motion would be equivalent to just a spherical motion, $\{S_o\}$.

Given a sphere - sphere contact, the relative motion between two spherical surfaces can be described by two spherical motions, a single spherical motion, or a revolute motion depending on if the surfaces are mobile, fixed, or one is mobile and one is fixed. Two spherical motions will occur if both spherical surfaces are mobile because neither

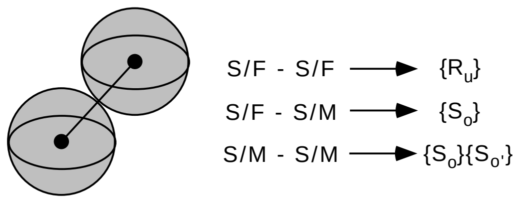

Fig. 3. A spherical surface to spherical surface contact

surface will know exactly where the contact occurred. A single spherical motion will occur if one of the spherical surfaces is mobile and one is fixed because one of the surfaces will know where the contact occurred relative to its coordinate system and the other surface will not. Therefore, the mobile surface can be rotated about any axis through the center of its spherical surface, and no change will be detected by either spherical surface. If both spherical surfaces are fixed, then the two surfaces can only rotate about an axis through the center of both spherical surfaces without a change being detected. This results in a revolute motion where the rotation axis goes through the centers of both spherical surfaces.

Figure 3 illustrated the three possible contacts between a sphere - sphere contact. Note, the case when both surfaces are mobile results in a spherical motion - spherical motion combination. This combination has only a dimension of five, not six. This is caused by a redundant motion in the combination. Both spherical joints can rotate about an axis through the center of both spherical surfaces. Therefore, only one of them is included in the group representation resulting in a decrease in the dimension for the combination.

Table III lists the group representation and dimension for the contacts listed in Table I. It should be noted that many of contacts share the same group representations.

IV. Analysis of Contact Combinations

With all of the contacts described by their respective group representation and dimension, it is possible to apply group operations to combinations of these contacts to find if they can be used to make a complete measurement of the relative position between the sensor frame and

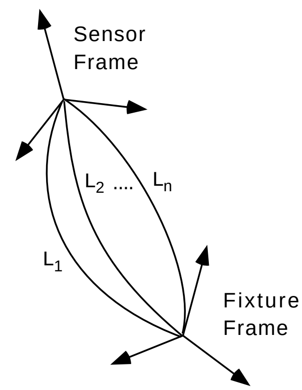

Fig. 4. Contact constraints between sensor frame and fixture frame

fixture frame or between two bodies. If we are given a set of contacts between the two frames and their respective group representations, then we know that the possible motion between the two frames is the intersection of the contact group representations. This concept is similar to the method Hervé used for analysis of parallel mechanism chains [3]. Ideally, we want to completely locate the fixture. Therefore, we want the dimension of this motion to be equal to the dimension of the tactile sensing features that form the fixture. Hence, we want to satisfy the following equation,

$$dim(\{L_1\} \cap \{L_2\} \cap \cdots \cap \{L_n\}) = dim(F), \qquad (2)$$

where $\{L_1\} \cdots \{L_n\}$ are the group representations for the contacts and F represents the geometric features of the fixture. Figure 4 shows the relationship between the frames and contact "constraints."

If a fixture is simply a mobile cylindrical surface, for example, then a complete reference frame can not be created by analysis of the features that compose the fixture. A cylindrical surface has a dimension of two due to rotations and translations along the center line of the surface. However, the location of the cylindrical surface can be found using various combinations of contacts. A collection of contacts to one cylindrical surface that results in a dimension of two should locate the fixture. If mobile line contacts are made (L/M - C/M contacts), then four lines will be necessary because a combination of four line-to-cylinder contacts results in a dimension of two. Figure 5 illustrates this ex-

TABLE III
The group representations and dimensions for the geometric contacts

Type of Contact	Group Representation	Dim.	Description
P/F - P/F	$\{S_o\}$	3	o is the contact pt.
P/F - S/F	$\{S_o\}$	3	o is the contact pt.
P/F - S/M	$\{S_o\}\{S_{o'}\}$	3+3-1=5	o is the contact pt., o' is the center of S/M
P/F - PL/F	$\{S_o\}$	3	o is the contact pt.
P/F - PL/M	$\{S_o\}\{T_P\}$	3+2=5	P is coincident with PL/M, o is the contact pt.
P/F - L/F	$\{S_o\}$	3	o is the contact pt.
P/F - L/M	$\{S_o\}\{T_u\}$	3+1=4	o is the contact pt., axis u goes through o
P/F - C/F	$\{S_o\}$	3	o is the contact pt.
P/F - C/M	$\{S_o\}\{C_u\}$	3+2=5	o is the contact pt., axis u is a dist. r away from o
S/F - S/F	$\{R_u\}$	1	axis u goes through both sphere centers
S/F - S/M	$\{S_o\}$	3	o is the center of the mobile sphere S/M
S/M - S/M	$\{S_o\}\{S_{o'}\}$	3+3-1=5	o and o' are the centers of the spheres
S/F - PL/F	$\{R_u\}$	1	$u \perp PL/F$ and goes through the center of S/F
S/F - PL/M	$\{G_P\}$	3	P is coincident with PL/M
S/M - PL/F	$\{S_o\}$	3	o is the center of the mobile sphere S/M
S/M - PL/M	$\{S_o\}\{T_P\}$	2+3=5	o is the center of S/M, P is coincident with PL/M
S/F - L/F	$\{R_u\}\{R_{u'}\}$	1+1=2	axis u goes through the center of S/F and the contact pt., axis u' is coincident with L/F
S/F - L/M	$\{R_u\}\{C_{u'}\}$	1+2=3	axis u goes through the center of S/F and the contact pt., axis u' is coincident with L/F
S/M - L/F	$\{S_o\}\{R_u\}$	3+1=4	o is the center of S/M, axis u is coincident with L/F
S/M - L/M	$\{S_o\}\{C_u\}$	3+2=5	o is the center of S/M, axis u is coincident with L/M
S/F - C/F	$\{R_u\}$	1	axis u goes through the center of S/F and the contact pt.
S/F - C/M	$\{R_u\}\{C_{u'}\}$	1+2=3	axis u goes through the center of S/F and the contact pt., and axis u' is coincident with the center line of C/M
S/M - C/F	$\{S_o\}$	3	o is the center of S/M
S/M - C/M	$\{S_o\}\{C_u\}$	3+2=5	o is the center of S/M, and axis u' is coincident with the center line of C/M
PL/F - PL/F	$\{G_P\}$	3	P is coincident with both planes
PL/F - L/F	$\{R_u\}$	1	axis u is coincident with L/F and on the plane
PL/M - L/F	$\{G_P\}\{R_u\}$	3+1=4	P is coincident with PL/M, and axis u is coincident with L/F and on P
PL/F - C/F	$\{T_u\}$	1	axis u is coincident with the center line of C/F
PL/F - C/M	$\{C_u\}$	2	axis u is coincident with the center line of C/M
PL/M - C/F	$\{G_P\}$	3	P is coincident with PL/M
PL/M - C/M	$\{G_P\}\{R_u\}$	3+1=4	axis u is coincident with the center line of C/M and, P is coincident with PL/M
L/F - L/F	$\{S_o\}$	3	o is the contact pt.
L/F - L/M	$\{S_o\}\{T_u\}$	3+1=4	o is the contact pt., and u is coincident with L/M
L/M - L/M	$\{C_u\}\{R_{u'}\}\{C_{u''}\}$	2+1+2=5	u is coincident with L/F #1, u'' is coincident with L/M #2, and $u' \perp$ the contact plane.
L/F - C/F	$\{R_u\}\{R_{u'}\}$	1+1=2	u is coincident with L/F, and u' is perpendicular to the contact plane.
L/F - C/M	$\{R_u\}\{R_{u'}\}\{C_{u''}\}$	1+1+2=4	u is coincident with L/F, u'' is coincident with the center line of C/M, and $u' \perp$ the contact plane.
L/M - C/F	$\{R_u\}\{C_{u'}\}$	1+2=3	u' is coincident with L/M, and $u' \perp$ the contact plane.
L/M - C/M	$\{C_u\}\{R_{u'}\}\{C_{u''}\}$	2+1+2=5	u is coincident with L/M, u'' is coincident with the center line of C/M, and $u' \perp$ the contact plane.
C/F - C/F	$\{R_u\}$	1	u is perpendicular to the contact plane
C/F - C/M	$\{R_u\}\{C_{u'}\}$	1+2=3	u' is coincident with the center line of C/M, and $u \perp$ the contact plane.
C/M - C/M	$\{C_u\}\{R_{u'}\}\{C_{u''}\}$	2+1+2=5	u and u'' are coincident with the center lines of C/M #1 and C/M #2, and $u' \perp$ the contact plane.

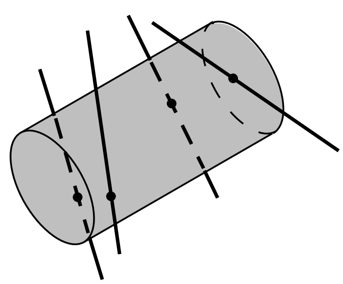

Fig. 5. A cylindrical mobile surface in contact with four mobile lines

ample. In this case the lines could be representing a line formed by a laser that has been barely broken by interference from the cylindrical surface indicating "contact" with the surface of the fixture.

It should be noted that the group representations are coordinate dependent. When finding the intersection of group representations between two bodies a coordinate system for each of the bodies must be chosen. Most of the group representations assume that the coordinate frames are located in a specific location. For example, any contact with a group representation $\{S_o\}$ assumes that both bodies have their respective coordinate systems located at the center of the spherical motion. Hence, when motion occurs, all displacements are part of the spherical motion group. If two contacts are used with spherical group representations, then only one can have the coordinate systems placed at its center; the other contact will no longer be represented by a spherical motion but a strange set of displacements (see Figure 6). However, changing the coordinate system

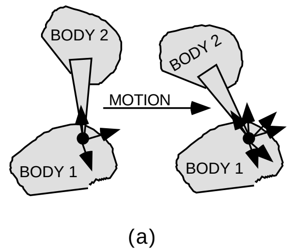

Fig. 6. Group representation dependence on the location of the coordinate system

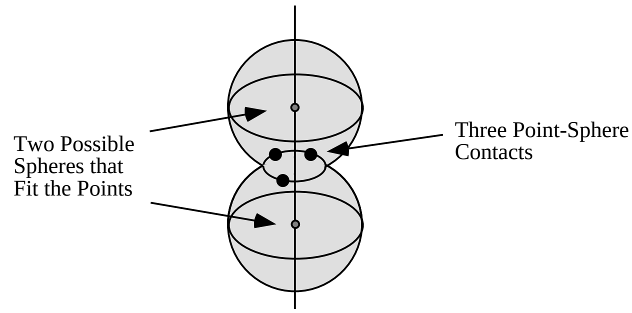

Fig. 7. Three points on a sphere gives two possible solutions

does not change the dimension of the motion caused by a geometric element contact.

Using equation 2, it is now possible to study contact combinations and determine if they can be used for measuring the relative position between the fixture and sensor frame. In other words, they can be used to determine appropriate contact combinations for locating a part referencing fixture. As discussed in an earlier example, if a cylinder is to be located in space, then all combinations to a cylindrical surface resulting in a combined dimension of two will locate the cylinder because a cylindrical surface has a dimension of two from rotation and translation along its center line.

For a complete reference measurement to be made, the dimension of the fixture's tactile sensing features should be zero. For most reference fixture applications a complete reference frame needs to be found. Hence, the combinations of contacts that result in a dimension of zero are important. More information on design of complete referencing fixtures (fixtures with a dimension of zero) can be found in Nederbragt and Ravani [9].

A shortcoming associated with equation 2 is that if a combination is found with dimension zero, it still may have a finite number of possible solutions. For example, the theory suggests that it takes three points to find the position of a sphere of known radius (a spherical surface has dimension three). However, if three points are used, two spheres will fit to the same three points (see Figure 7). In order to remedy this situation another piece of information is necessary. This will be the case with many of the combinations found.

Looking at Table III it is apparent that many of the geometric element contacts share the same group representations. This is due to the fact that a point is a special case of a mobile sphere of radius zero and a line is a cylinder of radius zero. Therefore, all of the possible contacts listed can be reduced from 41 cases to 15 (see Table IV).

Using equation 2 and the notation from Table IV, a list of all possible contact combinations that will result in a dimension of zero can be made. This is done by treating each contact as a constraint where the degree of constraint is six minus the degree of freedom. Using this paradigm, a complete measurement requires that the summation of the degrees of constraint for a contact combination set equal six or more. This works because the group representations, in general, do not share the same motions (i.e., the intersection of the representative group notations result in the identity motion only). This is not true for contacts represented by planar motion and spherical motion. Two spherical motions, two planar motions, and one planar motion/one spherical motion always share one degree of freedom; hence, this must be considered in the creation of contact combinations. A detailed list of these combinations is presented in the appendix.

V. Fixture Design Examples

The appendix lists 579 combination classes. If each class is broken down into contact combinations, then there are 17,460 different contact combinations that will, in general, result in a dimension of zero.

Because of the large number of contact combinations, it

TABLE IV
CLASSES OF CONTACTS

The Class Representation	The Group Representation	Contacts in the Class
$\{R\}$	$\{R_u\}$	S/F - S/F, S/F - PL/F, S/F - C/F, PL/F - L/F, C/F - C/F
$\{T_1\}$	$\{T_u\}$	PL/F - C/F
$\{R-R\}$	$\{R_u\}\{R_{u'}\}$	S/F - L/F, L/F - C/F
$\{C\}$	$\{C_u\}$	PL/F - C/M
$\{R-C\}$	$\{R_u\}\{C_{u'}\}$	S/F - L/M, S/F - C/M, L/M - C/F, C/F - C/M
$\{S\}$	$\{S_o\}$	P/F - P/F, P/F - S/F, P/F - PL/F, P/F - L/F, P/F - C/F, S/F - S/M, S/M - PL/F, S/M - C/F, L/F-L/F
$\{G_3\}$	$\{G_P\}$	S/F - PL/M, PL/F - PL/F, PL/M - C/M
$\{S-R\}$	$\{S_o\}\{R_u\}$	S/M - L/F
$\{S-T_1\}$	$\{S_o\}\{T_u\}$	P/F - L/M, L/F - L/M
$\{G_3-R\}$	$\{G_P\}\{R_u\}$	PL/M - L/F, PL/M - C/M
$\{R-R-C\}$	$\{R_u\}\{R_{u'}\}\{C_{u''}\}$	L/F - C/M
$\{S-S\}$	$\{S_o\}\{s_{o'}\}$	P/F - S/M, S/M - S/M
$\{S-C\}$	$\{S_o\}\{C_u\}$	P/F - C/M, S/M - L/M, S/M - C/M
$\{S-T_2\}$	$\{S_o\}\{T_P\}$	P/F - PL/M, S/M - PL/M
$\{C-R-C\}$	$\{C_u\}\{R_{u'}\}\{C_{u''}\}$	L/M - L/M, L/M - C/M, C/M - C/M

is difficult to assimilate all of the data and use it. Furthermore, not all of the contact combinations are practical. We have found that it is best to study subsets of the combination list. This allows for an in-depth study of particular combinations, which frequently results in the elimination of impractical combinations. We have already studied two subsets from the list: point to surface contact fixtures [8] and fixtures that include a plane of known location [10]. There are many more subsets that we intend to study. moreover, by creating the combination list and studying these subsets, we have discovered several combinations that have great potential for use as referencing fixtures. In fact, we have patented one of these [12].

Although the table can be very useful, the user should be aware of special cases. For example, if the contacts are made so that some of the contacts share geometric similarities (degenerate cases) that normally are not encountered, then additional contacts may become necessary. For example, three co-linear point-to-plane contacts are not sufficient for locating a plane; this is a degenerate case. The appendix only lists general combinations; hence, each specific case needs to be checked. Two specific examples are now given to illustrate the use of the appendix list.

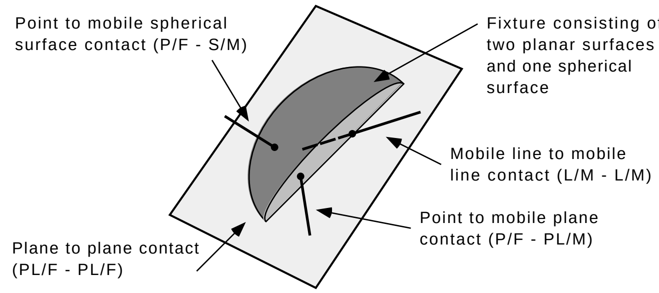

Fig. 8. Fixture example 1

A. Example One

For the first example, we will use the $\{G_3\}$\{S-S\}\{S-T_2\}\{C-R-C\} combination class (this is row number 224 of Table IV). There are 36 different combination sets in this class. The \{C-R-C\} contact set alone represents three available contacts, a L/M-L/M contact, a L/M-C/M contact, and a C/M-C/M contact. We need to choose one of the 36 different combination sets. For this example, we will use a PL/F-PL/F contact $\in \{G_3\}$ (plane to plane contact), a P/F-S/M contact $\in$ \{S-S\} (point to mobile sphere contact), a P/F-PL/M contact $\in$ \{S-T_2\} (point to mobile plane contact), and a L/M-L/M contact $\in$ \{C-R-C\} (mobile line to mobile line contact). These four contacts, when used together, result in a dimension of zero (the PL/F-PL/F contact reduces possible motion by three degrees of freedom, the other three contacts reduce the motion by one degree of freedom each). Hence, a fixture that uses these contacts can be located. In order to design a fixture using these contacts, the fixture must contain two planar surfaces (one for the $\{G_3\}$ contact and one for the \{S-T_2\} contact), one spherical surface (for the \{S-S\} contact), and one line for the \{C-R-C\} contact.

One possible design is a solid quarter sphere where the two planes that cut a sphere into a quarter of a sphere are touch sensing, in addition to the spherical surface and the line of intersection formed by the two planes. Figure 8 illustrates the use of this fixture design.

In this example, a finite number of solutions may be encountered. If this is the case, then additional contacts may be necessary to eliminate the incorrect solutions and

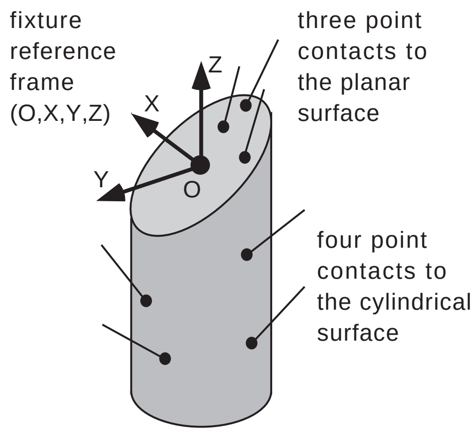

Fig. 9. Fixture example 2

determine the unique solution. This must be analyzed on a case by case basis.

B. Example Two

For the second example, we will use the {S-C}{S-C}{S-C}{S-T_2}{S-T_2}{S-T_2} combination class. We also will demonstrate how to construct a reference frame using these contacts. There are 40 different combination sets in this class. We are going to treat the three {S-T_2} classes as point-to-plane contacts (P/F - PL/M) and the three {S-C} classes as point-to-cylinder contacts. These six contacts, when used together, result in a dimension of zero (each contact eliminates one degree of motion). Hence, a fixture that uses these contacts can be located. In order to design a fixture using these contacts, the fixture must contain at least one planar surface (for the {S-T_2} contacts) and at least one cylindrical surface (for the {S-C} contacts).

We will use the simpliest design, one plane and one cylinder. Figure 9 illustrates one possible design of a plane/cylinder fixture. In this example, a finite number of solutions will be encountered because the cylindrical surface is quadratic; hence, we will use an additional point contact to the cylinder to gaurantee a unique solution. It may be possible to uniquely locate the fixture with just the six points, but this will require knowledge of the point trajectories prior to contact [8]. We will now give a detailed analysis to show how a unique reference frame can be created from our seven contacts.

Using our seven points, we need to create a reference frame consisting of three unique mutually perpendicular vectors and a unique origin. Four steps are necessary to create the frame. Step one, the normal to the planar surface needs to be found using the three point locations on this surface. Step two, a vector along the axis of the cylindrical surface needs to be found using the normal from step one and the four points on the cylindrical surface. Step three, using the vector along the axis of the cylindrical surface and the points on the cylindrical surface, a point on the axis needs to be found. Step four, using the results from the previous steps the reference frame needs to be created.

Let x_1, x_2, x_3, and x_4 be four point contact locations on the cylindrical surface. Let x_5, x_6, and x_7 be three points on the planar surface. Using points x_5, x_6, and x_7, we need to find the normal to the planar surface. This can easily be done by creating two vectors, $\vec{v}_1 = x_6 - x_5$ and $\vec{v}_2 = x_7 - x_5$, and taking the cross product between the two, $\vec{n} = \vec{v}_1 \times \vec{v}_2$ (see Figure 10). The equation for the plane can be calculated using $\vec{n}$ and a point on the planar surface (e.g., x_5).

With the normal $\vec{n}$ to the planar surface known, we can now find a vector along the axis of the cylindrical surface. Let $\vec{s} =< s_x, s_y, 1 >$ be this vector. The value of s_z is set to one to reduce the necessary computations. This can cause a problem if the axis of the cylinder is parallel with the x-y plane; however, this is very unlikely.

Using the four points on the cylindrical surface, we can create an equation with two unknowns, s_x and s_y. The derivation of the equation can be found in Nederbragt and Ravani [8]. The equation is:

$$\begin{bmatrix} s_x \; s_y \; 1 \end{bmatrix} \begin{bmatrix} n_{1_x} \; n_{2_x} \; n_{3_x} \\ n_{1_y} \; n_{2_y} \; n_{3_y} \\ n_{1_z} \; n_{2_z} \; n_{3_z} \end{bmatrix} \begin{bmatrix} ||\vec{p}_1 \times \vec{s}\,||^2 \\ ||\vec{p}_2 \times \vec{s}\,||^2 \\ ||\vec{p}_3 \times \vec{s}\,||^2 \end{bmatrix} = 0. \qquad (3)$$

where $\vec{p}_i = x_i - x_4$, $\vec{n}_1 = \vec{p}_2 \times \vec{p}_3$, $\vec{n}_2 = \vec{p}_3 \times \vec{p}_1$, and $\vec{n}_3 = \vec{p}_1 \times \vec{p}_2$. This cubic equation described the cylindrical surface in terms of the four points on its surface and the two unknowns. We have one equation and two unknowns, hence, we need another equation to find s_x and s_y.

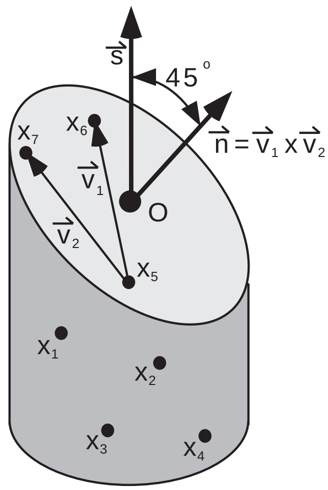

Fig. 10. Analysis of the coordinate frame attached to the plane-cylinder fixture

If the inclination of the plane is set to a specific angle (other than zero or ninety degrees) relative to the axis of the cylindrical surface, then, using vector algebra [4], another equation can be found. For simplicity we will let the angle between the axis of the cylindrical surface and the plane be 45 degrees. With the angle between the plane and axis equal to 45 degrees, the angle between $\vec{n}$ and $\vec{s}$ must also be 45 degrees (see Figure 10).

From vector algebra [4], the following equation (commonly referred to as the scalar product) is known:

$$\frac{\vec{k} \cdot \vec{l}}{||\vec{k}|| \; ||\vec{l}||} = \cos(\theta) \tag{4}$$

where θ is the angle between $\vec{k}$ and $\vec{l}$. Using equation 4, $\vec{s}$, $\vec{n}$ (which has been normalized), and the angle between them, the following equation is obtained,

$$\frac{\vec{n} \cdot \vec{s}}{||\vec{s}||} = \cos(45) = \frac{1}{\sqrt{2}}. \tag{5}$$

Taking the square of equation 5, we obtain

$$2(\vec{n} \cdot \vec{s})^2 = \vec{s} \cdot \vec{s}. \tag{6}$$

Expanding equation 6 using the components of $\vec{n}$ and $\vec{s}$, we get

$$2(n_x s_x + n_y s_y + n_z)^2 = s_x^2 + s_y^2 + s_z^2. \tag{7}$$

Equations 3 and 7 can be solved to find $\vec{s}$ using a standard mathematics package. Since equation 3 is cubic and equation 7 is quadratic, we will get up to six possible solutions for $\vec{s}$. These solutions represent different cylindrical surfaces with different radii that satisfy the equations. Since the radius of the fixture is known to the user, it can be used to eliminate the incorrect solutions [8]. With $\vec{s}$ known, we can now find a point, $q = (q_x, q_y, q_z)$, on the axis of the cylindrical surface using the method described in Nederbragt and Ravani [8]. With the position of both surfaces known it is now possible to create a reference frame.

Let $\vec{Z} = \vec{s}$, $\vec{Y} = \vec{n} \times \vec{s}$, and $\vec{X} = \vec{Y} \times \vec{Z}$ be our three unique mutually perpendicular reference frame vectors. Now we need an origin for the frame. The point where the axis of the cylindrical surface intersects the planar surface is unique and easily determinable, hence it would make an ideal origin. We denote this point as o. The axis of the cylindrical surface can be written as

$$(o_x, o_y, o_z) = (q_x, q_y, q_z) + l < s_x, s_y, s_z > \tag{8}$$

where $l \in R$. If equation 8 is substituted into the equation for the planar surface, then a value for l can be found that, when substituted back into equation 8, will determine the location of the point of intersection.

These examples represent just two of the 17,460 different contact combinations that are available in the appendix. Each one of the contact combinations can be made into one or more reference fixture designs.

VI. Conclusion

A method for type synthesis of geometric elements composed of contacts between points, lines, spherical surfaces, cylindrical surfaces, and planar surfaces is given. Using this method a list of contact combinations is derived that have the potential to measure the relative position between two bodies. Two examples are given that illustrate the use of this method. This method can be used for design of mechanical fixtures for part registration or referencing in robotics. The results can also have applications in grasp planning for determination of points or geometric features on a part for developing a stable grasp. The results together with evaluation of measurement errors and elimination of multiple solutions will be part of future extensions of this work.

Acknowledgment

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Appendix

Using the notation from Table IV, a listing of all possible contact combinations that will result in a dimension of zero is made, see Table V. This list gives the number and type of contacts to a reference fixture's body necessary for determination of a reference frame. Hence, it is now possible to determine if a given set of contacts to a fixture will work prior to doing the geometric calculations necessary to create a reference frame. As stated earlier, a finite number of possible positions may come out of the combination. These cases will require additional information for a unique solution to be found.

TABLE V

Enumeration of all possible contact combinations

Combination class	No.	Combination Class	No.	Combination class	No.
{R}{R}	15	{R-C}{S-R}{S-T_2}	8	{S}{G_3-R}{R-R-C}	18
{R}{T_1}	5	{R-C}{S-R}{C-R-C}	12	{S}{G_3-R}{S-S}	36
{R}{R-R}	10	{R-C}{S-T_1}{S-T_1}	12	{S}{G_3-R}{S-C}	54
{R}{C}	5	{R-C}{S-T_1}{G_3-R}	16	{S}{G_3-R}{S-T_2}	36
{R}{R-C}	20	{R-C}{S-T_1}{R-R-C}	8	{S}{G_3-R}{C-R-C}	54
{R}{S}	45	{R-C}{S-T_1}{S-S}	16	{S}{R-R-C}{R-R-C}	9
{R}{G_3}	15	{R-C}{S-T_1}{S-C}	24	{S}{R-R-C}{S-S}	18
{R}{S-R}	5	{R-C}{S-T_1}{S-T_2}	16	{S}{R-R-C}{S-C}	27
{R}{S-T_1}	10	{R-C}{S-T_1}{C-R-C}	24	{S}{R-R-C}{S-T_2}	18
{R}{G_3-R}	10	{R-C}{G_3-R}{G_3-R}	12	{S}{R-R-C}{C-R-C}	27
{R}{R-R-C}	5	{R-C}{G_3-R}{R-R-C}	8	{S}{S-S}{S-S}{S-S}	36
{R}{S-S}	10	{R-C}{G_3-R}{S-S}	16	{S}{S-S}{S-S}{S-C}	81
{R}{S-C}	15	{R-C}{G_3-R}{S-C}	24	{S}{S-S}{S-S}{S-T_2}	54
{R}{S-T_2}	10	{R-C}{G_3-R}{S-T_2}	16	{S}{S-S}{S-S}{C-R-C}	81
{R}{C-R-C}	15	{R-C}{G_3-R}{C-R-C}	24	{S}{S-S}{S-C}{S-C}	108
{T_1}{T_1}	1	{R-C}{R-R-C}{R-R-C}	4	{S}{S-S}{S-C}{S-T_2}	108
{T_1}{R-R}	2	{R-C}{R-R-C}{S-S}	8	{S}{S-S}{S-C}{C-R-C}	162
{T_1}{C}	1	{R-C}{R-R-C}{S-C}	12	{S}{S-S}{S-T_2}{S-T_2}	54
{T_1}{R-C}	4	{R-C}{R-R-C}{S-T_2}	8	{S}{S-S}{S-T_2}{C-R-C}	108
{T_1}{S}	9	{R-C}{R-R-C}{C-R-C}	12	{S}{S-S}{C-R-C}{C-R-C}	108
{T_1}{G_3}	3	{R-C}{S-S}{S-S}{S-S}	16	{S}{S-C}{S-C}{S-C}	90
{T_1}{S-R}	1	{R-C}{S-S}{S-S}{S-C}	36	{S}{S-C}{S-C}{S-T_2}	108
{T_1}{S-T_1}	2	{R-C}{S-S}{S-S}{S-T_2}	24	{S}{S-C}{S-C}{C-R-C}	162
{T_1}{G_3-R}	2	{R-C}{S-S}{S-S}{C-R-C}	36	{S}{S-C}{S-T_2}{S-T_2}	81
{T_1}{R-R-C}	1	{R-C}{S-S}{S-C}{S-C}	48	{S}{S-C}{S-T_2}{C-R-C}	162
{T_1}{S-S}	2	{R-C}{S-S}{S-C}{S-T_2}	48	{S}{S-C}{C-R-C}{C-R-C}	162
{T_1}{S-C}	3	{R-C}{S-S}{S-C}{C-R-C}	72	{S}{S-T_2}{S-T_2}{S-T_2}	36
{T_1}{S-T_2}	2	{R-C}{S-S}{S-T_2}{S-T_2}	24	{S}{S-T_2}{S-T_2}{C-R-C}	81
{T_1}{C-R-C}	3	{R-C}{S-S}{S-T_2}{C-R-C}	48	{S}{S-T_2}{C-R-C}{C-R-C}	108
{R-R}{R-R}	3	{R-C}{S-S}{C-R-C}{C-R-C}	48	{S}{C-R-C}{C-R-C}{C-R-C}	90
{R-R}{C}	2	{R-C}{S-C}{S-C}{S-C}	40	{G_3}{G_3}{G_3}	10
{R-R}{R-C}	8	{R-C}{S-C}{S-C}{S-T_2}	48	{G_3}{G_3}{S-R}	6
{R-R}{S}	18	{R-C}{S-C}{S-C}{C-R-C}	72	{G_3}{G_3}{S-T_1}	12
{R-R}{G_3}	6	{R-C}{S-C}{S-T_2}{S-T_2}	36	{G_3}{G_3}{G_3-R}	12
{R-R}{S-R}	2	{R-C}{S-C}{S-T_2}{C-R-C}	72	{G_3}{G_3}{R-R-C}	6
{R-R}{S-T_1}	4	{R-C}{S-C}{C-R-C}{C-R-C}	72	{G_3}{G_3}{S-S}	12
{R-R}{G_3-R}	4	{R-C}{S-T_2}{S-T_2}{S-T_2}	16	{G_3}{G_3}{S-C}	18
{R-R}{R-R-C}	2	{R-C}{S-T_2}{S-T_2}{C-R-C}	36	{G_3}{G_3}{S-T_2}	12
{R-R}{S-S}{S-S}	6	{R-C}{S-T_2}{C-R-C}{C-R-C}	48	{G_3}{G_3}{C-R-C}	18
{R-R}{S-S}{S-C}	12	{R-C}{C-R-C}{C-R-C}{C-R-C}	40	{G_3}{S-R}{S-R}	3
{R-R}{S-S}{S-T_2}	8	{S}{S}{S}	165	{G_3}{S-R}{S-T_1}	6
{R-R}{S-S}{C-R-C}	12	{S}{S}{G_3}	135	{G_3}{S-R}{G_3-R}	6
{R-R}{S-C}{S-C}	12	{S}{S}{S-R}	45	{G_3}{S-R}{R-R-C}	3
{R-R}{S-C}{S-T_2}	12	{S}{S}{S-T_1}	90	{G_3}{S-R}{S-S}	6
{R-R}{S-C}{C-R-C}	18	{S}{S}{G_3-R}	90	{G_3}{S-R}{S-C}	9
{R-R}{S-T_2}{S-T_2}	6	{S}{S}{R-R-C}	45	{G_3}{S-R}{S-T_2}	6
{R-R}{S-T_2}{C-R-C}	12	{S}{S}{S-S}	90	{G_3}{S-R}{C-R-C}	9
{R-R}{C-R-C}{C-R-C}	12	{S}{S}{S-C}	135	{G_3}{S-T_1}{S-T_1}	9
{C}{C}	1	{S}{S}{S-T_2}	90	{G_3}{S-T_1}{G_3-R}	12
{C}{R-C}	4	{S}{S}{C-R-C}	135	{G_3}{S-T_1}{R-R-C}	6
{C}{S}	9	{S}{G_3}{G_3}	54	{G_3}{S-T_1}{S-S}	12
{C}{G_3}	3	{S}{G_3}{S-R}	27	{G_3}{S-T_1}{S-C}	18
{C}{S-R}	1	{S}{G_3}{S-T_1}	54	{G_3}{S-T_1}{S-T_2}	12
{C}{S-T_1}	2	{S}{G_3}{G_3-R}	54	{G_3}{S-T_1}{C-R-C}	18
{C}{G_3-R}	2	{S}{G_3}{R-R-C}	27	{G_3}{G_3-R}{G_3-R}	9
{C}{R-R-C}	1	{S}{G_3}{S-S}	54	{G_3}{G_3-R}{R-R-C}	6
{C}{S-S}{S-S}	3	{S}{G_3}{S-C}	81	{G_3}{G_3-R}{S-S}	12
{C}{S-S}{S-C}	6	{S}{G_3}{S-T_2}	54	{G_3}{G_3-R}{S-C}	18
{C}{S-S}{S-T_2}	4	{S}{G_3}{C-R-C}	81	{G_3}{G_3-R}{S-T_2}	12
{C}{S-S}{C-R-C}	6	{S}{S-R}{S-R}	9	{G_3}{G_3-R}{C-R-C}	18
{C}{S-C}{S-C}	6	{S}{S-R}{S-T_1}	18	{G_3}{R-R-C}{R-R-C}	3
{C}{S-C}{S-T_2}	6	{S}{S-R}{G_3-R}	18	{G_3}{R-R-C}{S-S}	6
{C}{S-C}{C-R-C}	9	{S}{S-R}{R-R-C}	9	{G_3}{R-R-C}{S-C}	9
{C}{S-T_2}{S-T_2}	3	{S}{S-R}{S-S}	18	{G_3}{R-R-C}{S-T_2}	6
{C}{S-T_2}{C-R-C}	6	{S}{S-R}{S-C}	27	{G_3}{R-R-C}{C-R-C}	9
{C}{C-R-C}{C-R-C}	6	{S}{S-R}{S-T_2}	18	{G_3}{S-S}{S-S}{S-S}	12
{R-C}{R-C}	10	{S}{S-R}{C-R-C}	27	{G_3}{S-S}{S-S}{S-C}	27
{R-C}{S}	36	{S}{S-T_1}{S-T_1}	27	{G_3}{S-S}{S-S}{S-T_2}	18
{R-C}{G_3}	12	{S}{S-T_1}{G_3-R}	36	{G_3}{S-S}{S-S}{C-R-C}	27
{R-C}{S-R}{S-R}	4	{S}{S-T_1}{R-R-C}	18	{G_3}{S-S}{S-C}{S-C}	36
{R-C}{S-R}{S-T_1}	8	{S}{S-T_1}{S-S}	36	{G_3}{S-S}{S-C}{S-T_2}	36
{R-C}{S-R}{G_3-R}	8	{S}{S-T_1}{S-C}	54	{G_3}{S-S}{S-C}{C-R-C}	54
{R-C}{S-R}{R-R-C}	4	{S}{S-T_1}{S-T_2}	36	{G_3}{S-S}{S-T_2}{S-T_2}	18
{R-C}{S-R}{S-S}	8	{S}{S-T_1}{C-R-C}	54	{G_3}{S-S}{S-T_2}{C-R-C}	36
{R-C}{S-R}{S-C}	12	{S}{G_3-R}{G_3-R}	27	{G_3}{S-S}{C-R-C}{C-R-C}	36

Table Continues on Next Page

TABLE V

ENUMERATION OF ALL POSSIBLE CONTACT COMBINATIONS (CONTINUED)

Combination class	No.	Combination Class	No.	Combination class	No.
{G_3}{S-C}{S-C}{S-C}	30	{S-R}{S-S}{S-C}{C-R-C}{C-R-C}	36	{S-T_1}{S-S}{C-R-C}{C-R-C}{C-R-C}	40
{G_3}{S-C}{S-C}{S-T_2}	36	{S-R}{S-S}{S-T_2}{S-T_2}{S-T_2}	8	{S-T_1}{S-C}{S-C}{S-C}{S-C}	30
{G_3}{S-C}{S-C}{C-R-C}	54	{S-R}{S-S}{S-T_2}{S-T_2}{C-R-C}	18	{S-T_1}{S-C}{S-C}{S-C}{S-T_2}	40
{G_3}{S-C}{S-T_2}{S-T_2}	27	{S-R}{S-S}{S-T_2}{C-R-C}{C-R-C}	24	{S-T_1}{S-C}{S-C}{S-C}{C-R-C}	60
{G_3}{S-C}{S-T_2}{C-R-C}	54	{S-R}{S-S}{C-R-C}{C-R-C}{C-R-C}	20	{S-T_1}{S-C}{S-C}{S-T_2}{S-T_2}	36
{G_3}{S-C}{C-R-C}{C-R-C}	54	{S-R}{S-C}{S-C}{S-C}{S-C}	15	{S-T_1}{S-C}{S-C}{S-T_2}{C-R-C}	72
{G_3}{S-T_2}{S-T_2}{S-T_2}	12	{S-R}{S-C}{S-C}{S-C}{S-T_2}	20	{S-T_1}{S-C}{S-C}{C-R-C}{C-R-C}	72
{G_3}{S-T_2}{S-T_2}{C-R-C}	27	{S-R}{S-C}{S-C}{S-C}{C-R-C}	30	{S-T_1}{S-C}{S-T_2}{S-T_2}{S-T_2}	24
{G_3}{S-T_2}{C-R-C}{C-R-C}	36	{S-R}{S-C}{S-C}{S-T_2}{S-T_2}	18	{S-T_1}{S-C}{S-T_2}{S-T_2}{C-R-C}	54
{G_3}{C-R-C}{C-R-C}{C-R-C}	30	{S-R}{S-C}{S-C}{S-T_2}{C-R-C}	36	{S-T_1}{S-C}{S-T_2}{C-R-C}{C-R-C}	72
{S-R}{S-R}{S-R}	1	{S-R}{S-C}{S-C}{C-R-C}{C-R-C}	36	{S-T_1}{S-C}3x{C-R-C}	60
{S-R}{S-R}{S-T_1}	2	{S-R}{S-C}{S-T_2}{S-T_2}{S-T_2}	12	{S-T_1}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	10
{S-R}{S-R}{G_3-R}	2	{S-R}{S-C}{S-T_2}{S-T_2}{C-R-C}	27	{S-T_1}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	24
{S-R}{S-R}{R-R-C}	1	{S-R}{S-C}{S-T_2}{C-R-C}{C-R-C}	36	{S-T_1}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	36
{S-R}{S-R}{S-S}{S-S}	3	{S-R}{S-C}{C-R-C}{C-R-C}{C-R-C}	30	{S-T_1}{S-T_2}3x{C-R-C}	40
{S-R}{S-R}{S-S}{S-C}	6	{S-R}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	5	{S-T_1}4x{C-R-C}	30
{S-R}{S-R}{S-S}{S-T_2}	4	{S-R}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	12	{G_3-R}{G_3-R}{G_3-R}	4
{S-R}{S-R}{S-S}{C-R-C}	6	{S-R}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	18	{G_3-R}{G_3-R}{R-R-C}	3
{S-R}{S-R}{S-C}{S-C}	6	{S-R}{S-T_2}3x{C-R-C}	20	{G_3-R}{G_3-R}{S-S}{S-S}	9
{S-R}{S-R}{S-C}{S-T_2}	6	{S-R}4x{C-R-C}	15	{G_3-R}{G_3-R}{S-S}{S-C}	18
{S-R}{S-R}{S-C}{C-R-C}	9	{S-T_1}{S-T_1}{S-T_1}	4	{G_3-R}{G_3-R}{S-S}{S-T_2}	12
{S-R}{S-R}{S-T_2}{S-T_2}	3	{S-T_1}{S-T_1}{G_3-R}	6	{G_3-R}{G_3-R}{S-S}{C-R-C}	18
{S-R}{S-R}{S-T_2}{C-R-C}	6	{S-T_1}{S-T_1}{R-R-C}	3	{G_3-R}{G_3-R}{S-C}{S-C}	18
{S-R}{S-R}{C-R-C}{C-R-C}	6	{S-T_1}{S-T_1}{S-S}{S-S}	9	{G_3-R}{G_3-R}{S-C}{S-T_2}	18
{S-R}{S-T_1}{S-T_1}	3	{S-T_1}{S-T_1}{S-S}{S-C}	18	{G_3-R}{G_3-R}{S-C}{C-R-C}	27
{S-R}{S-T_1}{G_3-R}	4	{S-T_1}{S-T_1}{S-S}{S-T_2}	12	{G_3-R}{G_3-R}{S-T_2}{S-T_2}	9
{S-R}{S-T_1}{R-R-C}	2	{S-T_1}{S-T_1}{S-S}{C-R-C}	18	{G_3-R}{G_3-R}{S-T_2}{C-R-C}	18
{S-R}{S-T_1}{S-S}{S-S}	6	{S-T_1}{S-T_1}{S-C}{S-C}	18	{G_3-R}{G_3-R}{C-R-C}{C-R-C}	18
{S-R}{S-T_1}{S-S}{S-C}	12	{S-T_1}{S-T_1}{S-C}{S-T_2}	18	{G_3-R}{R-R-C}{R-R-C}	2
{S-R}{S-T_1}{S-S}{S-T_2}	8	{S-T_1}{S-T_1}{S-C}{C-R-C}	27	{G_3-R}{R-R-C}{S-S}{S-S}	6
{S-R}{S-T_1}{S-S}{C-R-C}	12	{S-T_1}{S-T_1}{S-T_2}{S-T_2}	9	{G_3-R}{R-R-C}{S-S}{S-C}	12
{S-R}{S-T_1}{S-C}{S-C}	12	{S-T_1}{S-T_1}{S-T_2}{C-R-C}	18	{G_3-R}{R-R-C}{S-S}{S-T_2}	8
{S-R}{S-T_1}{S-C}{S-T_2}	12	{S-T_1}{S-T_1}{C-R-C}{C-R-C}	18	{G_3-R}{R-R-C}{S-S}{C-R-C}	12
{S-R}{S-T_1}{S-C}{C-R-C}	18	{S-T_1}{G_3-R}{G_3-R}	6	{G_3-R}{R-R-C}{S-C}{S-C}	12
{S-R}{S-T_1}{S-T_2}{S-T_2}	6	{S-T_1}{G_3-R}{R-R-C}	4	{G_3-R}{R-R-C}{S-C}{S-T_2}	12
{S-R}{S-T_1}{S-T_2}{C-R-C}	12	{S-T_1}{G_3-R}{S-S}{S-S}	12	{G_3-R}{R-R-C}{S-C}{C-R-C}	18
{S-R}{S-T_1}{C-R-C}{C-R-C}	12	{S-T_1}{G_3-R}{S-S}{S-C}	24	{G_3-R}{R-R-C}{S-T_2}{S-T_2}	6
{S-R}{G_3-R}{G_3-R}	3	{S-T_1}{G_3-R}{S-S}{S-T_2}	16	{G_3-R}{R-R-C}{S-T_2}{C-R-C}	12
{S-R}{G_3-R}{R-R-C}	2	{S-T_1}{G_3-R}{S-S}{C-R-C}	24	{G_3-R}{R-R-C}{C-R-C}{C-R-C}	12
{S-R}{G_3-R}{S-S}{S-S}	6	{S-T_1}{G_3-R}{S-C}{S-C}	24	{G_3-R}{S-S}{S-S}{S-S}{S-S}	10
{S-R}{G_3-R}{S-S}{S-C}	12	{S-T_1}{G_3-R}{S-C}{S-T_2}	24	{G_3-R}{S-S}{S-S}{S-S}{S-C}	24
{S-R}{G_3-R}{S-S}{S-T_2}	8	{S-T_1}{G_3-R}{S-C}{C-R-C}	36	{G_3-R}{S-S}{S-S}{S-S}{S-T_2}	16
{S-R}{G_3-R}{S-S}{C-R-C}	12	{S-T_1}{G_3-R}{S-T_2}{S-T_2}	12	{G_3-R}{S-S}{S-S}{S-S}{C-R-C}	24
{S-R}{G_3-R}{S-C}{S-C}	12	{S-T_1}{G_3-R}{S-T_2}{C-R-C}	24	{G_3-R}{S-S}{S-S}{S-C}{S-C}	36
{S-R}{G_3-R}{S-C}{S-T_2}	12	{S-T_1}{G_3-R}{C-R-C}{C-R-C}	24	{G_3-R}{S-S}{S-S}{S-C}{S-T_2}	36
{S-R}{G_3-R}{S-C}{C-R-C}	18	{S-T_1}{R-R-C}{R-R-C}	2	{G_3-R}{S-S}{S-S}{S-C}{C-R-C}	54
{S-R}{G_3-R}{S-T_2}{S-T_2}	6	{S-T_1}{R-R-C}{S-S}{S-S}	6	{G_3-R}{S-S}{S-S}{S-T_2}{S-T_2}	18
{S-R}{G_3-R}{S-T_2}{C-R-C}	12	{S-T_1}{R-R-C}{S-S}{S-C}	12	{G_3-R}{S-S}{S-S}{S-T_2}{C-R-C}	36
{S-R}{G_3-R}{C-R-C}{C-R-C}	12	{S-T_1}{R-R-C}{S-S}{S-T_2}	8	{G_3-R}{S-S}{S-S}{C-R-C}{C-R-C}	36
{S-R}{R-R-C}{R-R-C}	1	{S-T_1}{R-R-C}{S-S}{C-R-C}	12	{G_3-R}{S-S}{S-C}{S-C}{S-C}	40
{S-R}{R-R-C}{S-S}{S-S}	3	{S-T_1}{R-R-C}{S-C}{S-C}	12	{G_3-R}{S-S}{S-C}{S-C}{S-T_2}	48
{S-R}{R-R-C}{S-S}{S-C}	6	{S-T_1}{R-R-C}{S-C}{S-T_2}	12	{G_3-R}{S-S}{S-C}{S-C}{C-R-C}	72
{S-R}{R-R-C}{S-S}{S-T_2}	4	{S-T_1}{R-R-C}{S-C}{C-R-C}	18	{G_3-R}{S-S}{S-C}{S-T_2}{S-T_2}	36
{S-R}{R-R-C}{S-S}{C-R-C}	6	{S-T_1}{R-R-C}{S-T_2}{S-T_2}	6	{G_3-R}{S-S}{S-C}{S-T_2}{C-R-C}	72
{S-R}{R-R-C}{S-C}{S-C}	6	{S-T_1}{R-R-C}{S-T_2}{C-R-C}	12	{G_3-R}{S-S}{S-C}{C-R-C}{C-R-C}	72
{S-R}{R-R-C}{S-C}{S-T_2}	6	{S-T_1}{R-R-C}{C-R-C}{C-R-C}	12	{G_3-R}{S-S}{S-T_2}{S-T_2}{S-T_2}	16
{S-R}{R-R-C}{S-C}{C-R-C}	9	{S-T_1}{S-S}{S-S}{S-S}{S-S}	10	{G_3-R}{S-S}{S-T_2}{S-T_2}{C-R-C}	36
{S-R}{R-R-C}{S-T_2}{S-T_2}	3	{S-T_1}{S-S}{S-S}{S-S}{S-C}	24	{G_3-R}{S-S}{S-T_2}{C-R-C}{C-R-C}	48
{S-R}{R-R-C}{S-T_2}{C-R-C}	6	{S-T_1}{S-S}{S-S}{S-S}{S-T_2}	16	{G_3-R}{S-S}3x{C-R-C}	40
{S-R}{R-R-C}{C-R-C}{C-R-C}	6	{S-T_1}{S-S}{S-S}{S-S}{C-R-C}	24	{G_3-R}{S-C}{S-C}{S-C}{S-C}	30
{S-R}{S-S}{S-S}{S-S}{S-S}	5	{S-T_1}{S-S}{S-S}{S-C}{S-C}	36	{G_3-R}{S-C}{S-C}{S-C}{S-T_2}	40
{S-R}{S-S}{S-S}{S-S}{S-C}	12	{S-T_1}{S-S}{S-S}{S-C}{S-T_2}	36	{G_3-R}{S-C}{S-C}{S-C}{C-R-C}	60
{S-R}{S-S}{S-S}{S-S}{S-T_2}	8	{S-T_1}{S-S}{S-S}{S-C}{C-R-C}	54	{G_3-R}{S-C}{S-C}{S-T_2}{S-T_2}	36
{S-R}{S-S}{S-S}{S-S}{C-R-C}	12	{S-T_1}{S-S}{S-S}{S-T_2}{S-T_2}	18	{G_3-R}{S-C}{S-C}{S-T_2}{C-R-C}	72
{S-R}{S-S}{S-S}{S-C}{S-C}	18	{S-T_1}{S-S}{S-S}{S-T_2}{C-R-C}	36	{G_3-R}{S-C}{S-C}{C-R-C}{C-R-C}	72
{S-R}{S-S}{S-S}{S-C}{S-T_2}	18	{S-T_1}{S-S}{S-S}{C-R-C}{C-R-C}	36	{G_3-R}{S-C}{S-T_2}{S-T_2}{S-T_2}	24
{S-R}{S-S}{S-S}{S-C}{C-R-C}	27	{S-T_1}{S-S}{S-C}{S-C}{S-C}	40	{G_3-R}{S-C}{S-T_2}{S-T_2}{C-R-C}	54
{S-R}{S-S}{S-S}{S-T_2}{S-T_2}	9	{S-T_1}{S-S}{S-C}{S-C}{S-T_2}	48	{G_3-R}{S-C}{S-T_2}{C-R-C}{C-R-C}	72
{S-R}{S-S}{S-S}{S-T_2}{C-R-C}	18	{S-T_1}{S-S}{S-C}{S-C}{C-R-C}	72	{G_3-R}{S-C}{C-R-C}{C-R-C}{C-R-C}	60
{S-R}{S-S}{S-S}2x{C-R-C}	18	{S-T_1}{S-S}{S-C}{S-T_2}{S-T_2}	36	{G_3-R}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	10
{S-R}{S-S}{S-C}{S-C}{S-C}	20	{S-T_1}{S-S}{S-C}{S-T_2}{C-R-C}	72	{G_3-R}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	24
{S-R}{S-S}{S-C}{S-C}{S-T_2}	24	{S-T_1}{S-S}{S-C}{C-R-C}{C-R-C}	72	{G_3-R}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	36
{S-R}{S-S}{S-C}{S-C}{C-R-C}	36	{S-T_1}{S-S}{S-T_2}{S-T_2}{S-T_2}	16	{G_3-R}{S-T_2}3x{C-R-C}	40
{S-R}{S-S}{S-C}{S-T_2}{S-T_2}	18	{S-T_1}{S-S}{S-T_2}{S-T_2}{C-R-C}	36	{G_3-R}4x{C-R-C}	30
{S-R}{S-S}{S-C}{S-T_2}{C-R-C}	36	{S-T_1}{S-S}{S-T_2}{C-R-C}{C-R-C}	48	{R-R-C}{R-R-C}{R-R-C}	1

Table Continues on Next Page

TABLE V

Enumeration of all possible contact combinations (continued)

Combination class	No.	Combination Class	No.
{R-R-C}{R-R-C}{S-S}{S-S}	3	{S-S}{S-S}{S-C}{S-C}{S-C}{S-C}	45
{R-R-C}{R-R-C}{S-S}{S-C}	6	{S-S}{S-S}{S-C}{S-C}{S-C}{S-T_2}	60
{R-R-C}{R-R-C}{S-S}{S-T_2}	4	{S-S}{S-S}{S-C}{S-C}{S-C}{C-R-C}	90
{R-R-C}{R-R-C}{S-S}{C-R-C}	6	{S-S}{S-S}{S-C}{S-C}{S-T_2}{S-T_2}	54
{R-R-C}{R-R-C}{S-C}{S-C}	6	{S-S}{S-S}{S-C}{S-C}{S-T_2}{C-R-C}	108
{R-R-C}{R-R-C}{S-C}{S-T_2}	6	{S-S}{S-S}{S-C}{S-C}{C-R-C}{C-R-C}	108
{R-R-C}{R-R-C}{S-C}{C-R-C}	9	{S-S}{S-S}{S-C}{S-T_2}{S-T_2}{S-T_2}	36
{R-R-C}{R-R-C}{S-T_2}{S-T_2}	3	{S-S}{S-S}{S-C}{S-T_2}{S-T_2}{C-R-C}	81
{R-R-C}{R-R-C}{S-T_2}{C-R-C}	6	{S-S}{S-S}{S-C}{S-T_2}{C-R-C}{C-R-C}	108
{R-R-C}{R-R-C}{C-R-C}{C-R-C}	6	{S-S}{S-S}{S-C}{C-R-C}{C-R-C}{C-R-C}	90
{R-R-C}{S-S}{S-S}{S-S}{S-S}	5	{S-S}{S-S}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	15
{R-R-C}{S-S}{S-S}{S-S}{S-C}	12	{S-S}{S-S}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	36
{R-R-C}{S-S}{S-S}{S-S}{S-T_2}	8	{S-S}{S-S}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	54
{R-R-C}{S-S}{S-S}{S-S}{C-R-C}	12	{S-S}{S-S}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	60
{R-R-C}{S-S}{S-S}{S-C}{S-C}	18	{S-S}{S-S}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	45
{R-R-C}{S-S}{S-S}{S-C}{S-T_2}	18	{S-S}{S-C}{S-C}{S-C}{S-C}{S-C}	42
{R-R-C}{S-S}{S-S}{S-C}{C-R-C}	27	{S-S}{S-C}{S-C}{S-C}{S-C}{S-T_2}	60
{R-R-C}{S-S}{S-S}{S-T_2}{S-T_2}	9	{S-S}{S-C}{S-C}{S-C}{S-C}{C-R-C}	90
{R-R-C}{S-S}{S-S}{S-T_2}{C-R-C}	18	{S-S}{S-C}{S-C}{S-C}{S-T_2}{S-T_2}	60
{R-R-C}{S-S}{S-S}{C-R-C}{C-R-C}	18	{S-S}{S-C}{S-C}{S-C}{S-T_2}{C-R-C}	120
{R-R-C}{S-S}{S-C}{S-C}{S-C}	20	{S-S}{S-C}{S-C}{S-C}{C-R-C}{C-R-C}	120
{R-R-C}{S-S}{S-C}{S-C}{S-T_2}	24	{S-S}{S-C}{S-C}{S-T_2}{S-T_2}{S-T_2}	48
{R-R-C}{S-S}{S-C}{S-C}{C-R-C}	36	{S-S}{S-C}{S-C}{S-T_2}{S-T_2}{C-R-C}	108
{R-R-C}{S-S}{S-C}{S-T_2}{S-T_2}	18	{S-S}{S-C}{S-C}{S-T_2}{C-R-C}{C-R-C}	144
{R-R-C}{S-S}{S-C}{S-T_2}{C-R-C}	36	{S-S}{S-C}{S-C}{C-R-C}{C-R-C}{C-R-C}	120
{R-R-C}{S-S}{S-C}{C-R-C}{C-R-C}	36	{S-S}{S-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	30
{R-R-C}{S-S}{S-T_2}{S-T_2}{S-T_2}	8	{S-S}{S-C}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	72
{R-R-C}{S-S}{S-T_2}{S-T_2}{C-R-C}	18	{S-S}{S-C}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	108
{R-R-C}{S-S}{S-T_2}{C-R-C}{C-R-C}	24	{S-S}{S-C}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	120
{R-R-C}{S-S}{C-R-C}{C-R-C}{C-R-C}	20	{S-S}{S-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	90
{R-R-C}{S-C}{S-C}{S-C}{S-C}	15	{S-S}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	12
{R-R-C}{S-C}{S-C}{S-C}{S-T_2}	20	{S-S}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	30
{R-R-C}{S-C}{S-C}{S-C}{C-R-C}	30	{S-S}{S-T_2}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	48
{R-R-C}{S-C}{S-C}{S-T_2}{S-T_2}	18	{S-S}{S-T_2}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	60
{R-R-C}{S-C}{S-C}{S-T_2}{C-R-C}	36	{S-S}{S-T_2}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	60
{R-R-C}{S-C}{S-C}{C-R-C}{C-R-C}	36	{S-S}{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	42
{R-R-C}{S-C}{S-T_2}{S-T_2}{S-T_2}	12	{S-C}{S-C}{S-C}{S-C}{S-C}{S-C}	28
{R-R-C}{S-C}{S-T_2}{S-T_2}{C-R-C}	27	{S-C}{S-C}{S-C}{S-C}{S-C}{S-T_2}	42
{R-R-C}{S-C}{S-T_2}{C-R-C}{C-R-C}	36	{S-C}{S-C}{S-C}{S-C}{S-C}{C-R-C}	63
{R-R-C}{S-C}{C-R-C}{C-R-C}{C-R-C}	30	{S-C}{S-C}{S-C}{S-C}{S-T_2}{S-T_2}	45
{R-R-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	5	{S-C}{S-C}{S-C}{S-C}{S-T_2}{C-R-C}	90
{R-R-C}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	12	{S-C}{S-C}{S-C}{S-C}{C-R-C}{C-R-C}	90
{R-R-C}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	18	{S-C}{S-C}{S-C}{S-T_2}{S-T_2}{S-T_2}	40
{R-R-C}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	20	{S-C}{S-C}{S-C}{S-T_2}{S-T_2}{C-R-C}	90
{R-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	15	{S-C}{S-C}{S-C}{S-T_2}{C-R-C}{C-R-C}	120
{S-S}{S-S}{S-S}{S-S}{S-S}{S-S}	7	{S-C}{S-C}{S-C}{C-R-C}{C-R-C}{C-R-C}	100
{S-S}{S-S}{S-S}{S-S}{S-S}{S-C}	18	{S-C}{S-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	30
{S-S}{S-S}{S-S}{S-S}{S-S}{S-T_2}	12	{S-C}{S-C}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	72
{S-S}{S-S}{S-S}{S-S}{S-S}{C-R-C}	18	{S-C}{S-C}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	108
{S-S}{S-S}{S-S}{S-S}{S-C}{S-C}	30	{S-C}{S-C}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	120
{S-S}{S-S}{S-S}{S-S}{S-C}{S-T_2}	30	{S-C}{S-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	90
{S-S}{S-S}{S-S}{S-S}{S-C}{C-R-C}	45	{S-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	18
{S-S}{S-S}{S-S}{S-S}{S-T_2}{S-T_2}	15	{S-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	45
{S-S}{S-S}{S-S}{S-S}{S-T_2}{C-R-C}	30	{S-C}{S-T_2}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	72
{S-S}{S-S}{S-S}{S-S}{C-R-C}{C-R-C}	30	{S-C}{S-T_2}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	90
{S-S}{S-S}{S-S}{S-C}{S-C}{S-C}	40	{S-C}{S-T_2}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	90
{S-S}{S-S}{S-S}{S-C}{S-C}{S-T_2}	48	{S-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	63
{S-S}{S-S}{S-S}{S-C}{S-C}{C-R-C}	72	{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	7
{S-S}{S-S}{S-S}{S-C}{S-T_2}{S-T_2}	36	{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	18
{S-S}{S-S}{S-S}{S-C}{S-T_2}{C-R-C}	72	{S-T_2}{S-T_2}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	30
{S-S}{S-S}{S-S}{S-C}{C-R-C}{C-R-C}	72	{S-T_2}{S-T_2}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	40
{S-S}{S-S}{S-S}{S-T_2}{S-T_2}{S-T_2}	16	{S-T_2}{S-T_2}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	45
{S-S}{S-S}{S-S}{S-T_2}{S-T_2}{C-R-C}	36	{S-T_2}{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	42
{S-S}{S-S}{S-S}{S-T_2}{C-R-C}{C-R-C}	48	{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	28
{S-S}{S-S}{S-S}{C-R-C}{C-R-C}{C-R-C}	40		
Number of combination classes = 579			
Number of combinations = 17,460			