# Computational Issues in the Kinematic Design of Tactile Sensing Fixtures

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**Summary.** This paper discusses computational issues in kinematic design of tactile sensing fixtures used in robotics applications. It deals with mechanical fixtures built or modeled by feature surfaces consisting of planes, spheres, and cylinders. It develops the governing equations for locating each of these geometric objects using tactile sensing probes. It shows that although four points are needed to locate a sphere, in many applications sensing three points is sufficient for referencing. In the case of a cylinder it is shown that in general six points are necessary and that in many applications five points are sufficient for locating the cylinder. The paper reduces the governing equations for a cylinder to a set of polynomial equations consisting of a second-degree and a third-degree equation. The solutions of this set are found using symbolic computations. The results are applied to the kinematic design and analysis of a mechanical fixture consisting of a sphere and a cylinder as its feature surfaces.

**Keywords.** Fixture, tactile sensing, robotics, calibration, part-referencing.

## 1. Introduction

Tactile sensing fixtures are used in robotics and manufacturing – see, for example, Duffie et al. (1984), McCallion and Pham (1984), Mooring and Pack (1987), Ravani and Ge (1991), and Slocum (1988) – for part referencing and calibration. This is the process of determining the relative location of a part with respect to a tool (such as a machine tool, a robot, or a material-handling system) or with respect to a world coordinate system. In robot calibration, the position of the end-effector is usually measured at a set of predetermined locations using some form of a sensing system. This data is then combined with joint-encoder readings from the same set of locations to update the kinematic parameters of the robot in its programming system – see, for example, Roth, Mooring and Ravani (1987) or Hollerbach (1988) – to improve its positioning accuracy. Since both part-referencing and calibration require measurements of relative locations between two objects, tactile-sensing mechanical fixtures are generally used to simplify the sensing function and to improve repeatability.

In order for a tactile sensing mechanical fixtures to be useful, a reference frame must be created out of the geometric contacts made to the surface of the fixture. Therefore, the geometric feature surfaces and contact conditions are critical to the design and operation of a practical fixture. This chapter deals with computational

aspects of localization of feature surfaces using the contact conditions. In particular, we develop methods for dealing with feature surfaces consisting of planes, spheres, and cylinders. The results for planes and spheres (in the general case) are known in some form or another, but the results for cylinder are new as well as some special cases for spheres.

The work presented in this paper addresses the computational aspect of some of our recent work (Nederbragt and Ravani, 1996 and 1997) in developing a scientific foundation for the kinematic design of mechanical fixtures.

The organization of the chapter is as follows: First, we discuss part-referencing based on touch-sensing. We then give methods for finding a plane, a sphere, and a cylinder in three-dimensional Euclidean space using a finite number of points on the surface of these geometric elements. Finally, learning from our computational analysis methods, we apply these methods to create a reference frame for a simple cylinder-sphere reference fixture.

## 2. Part-Referencing Based on Touch-Sensing

Part-referencing using tactile sensing involves bringing a sensing element into contact with a surface of the part, activating the touch sensor, and measuring the location of the touch point in the sensor coordinate system. This was the case, for example, in the system described by Duffie et al. (1984) where a touch sensor was attached to the end of a robot and it was moved until it contacted the spherical surface of a fixture. In such a system the location of the touch is only known in the robot manipulator frame. The shape of the touch surface on the fixture is, however, completely known. (In this case it is a sphere.)

If several touches are made to the surface, then enough information may be obtained to determine the relative location of the two frames. Duffie et al. (1984) used a fixture consisting of three separate spheres of known radii. They found that four separate touches to each of the spheres made it possible to determine the location of the fixture with respect to the robot. McCallion and Pham (1984) used three non-collinear touches to a plane to determine its location in space, and, using three perpendicular planes of a cube, the relative location of the robot to the fixture was found. Nederbragt and Ravani (1996, 1997) developed a scientific framework for design of such fixtures. In their work they created several new fixture geometries and also enumerated the geometric contact combinations necessary for the elimination of all continuous motions.

With the introduction of many new possible fixture geometries, the next step is to find methods for localizing the feature geometries in space based on the contact conditions. This may require the solution of several non-linear equations using computational methods. In the next sections, methods for determining the location of a plane, a sphere, and a cylinder are given. In the case of the plane, the methods for finding their location are simple and well known (McCallion and Pham, 1984), but for completeness a common method is given. The sphere case is also well known. However, we have developed a method that can reduce the number of points necessary for the determination of the location of the sphere in certain instances. A method for locating a cylinder is also given. The cylinder case is significantly more complicated than the sphere or the plane and, therefore, is the main focus of this paper. The method for finding its location has applications in fixture design, coordinate-measuring machines (CMM) and the Burmester theory for the design of linkages.

The three feature surfaces of planes, spheres and cylinders are the building blocks for a large number of practical fixture geometries.

## 3. Determination of the Location of a Plane Using Three Points

It is well known that the location of a plane can be found in three-dimensional Euclidean space if the location of three points on the plane are known (McCallion and Pham, 1984) (see Figure 1). However, for completeness, a common method is given.

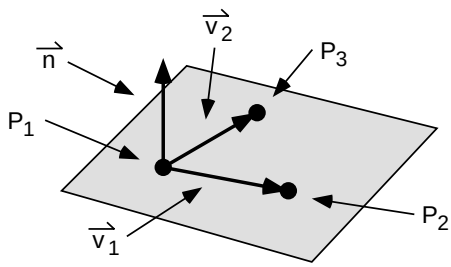


**Fig. 1.** Three points on a plane

Let points $p_1$, $p_2$, and $p_3$ be on a plane where $p_i = (x_i, y_i, z_i)$. We now define two vectors as $\mathbf{v}_1 = p_2 - p_1$ and $\mathbf{v}_2 = p_3 - p_1$. Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (a, b, c)$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel to the plane, $\mathbf{n}$ must be normal to the plane. From algebraic geometry (1939), the equation of the plane is:

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \tag{1}$$

or, in general form

$$ax + by + cz + d = 0 \tag{2}$$

where $d = -ax_1 - by_1 - cz_1$.

## 4. Determination of the Location of a Sphere Using Three Points

In general, it takes four points on the surface of a sphere to determine the radius and the center of that sphere (Duffie et al., 1984). This is well known. However, if the radius of the sphere is already known (this is always the case for referencing fixtures), then three points on the surface of the sphere will give two possible solutions (see Figure 2). If a touch probe is being used to make the point contacts to the sphere, then the orientation of the probe during contact can be used to possibly eliminate the incorrect sphere out of the two possible solutions. This makes the use of spheres for referencing more appealing because a three-sphere fixture may require significantly less than the twelve contacts used by Duffie et al. (1984) to make a complete reference measurement.

An algebraic method is given in the next section that will determine the location of a sphere of known radius. An example is also given that uses this algebraic method.

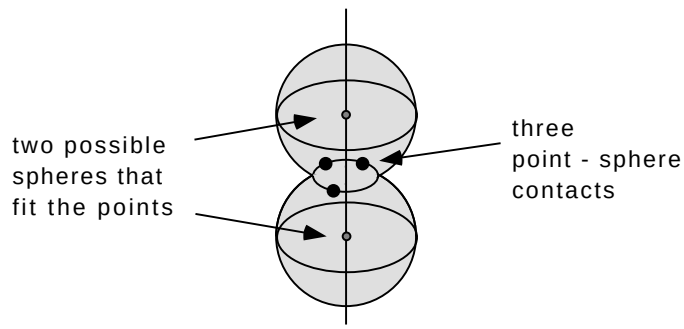


**Fig. 2.** Three points on a sphere gives two possible solutions

### 4.1 Algebraic Method

The general equation for a sphere in space is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 - r^2 = 0, \tag{3}$$

where $(a, b, c)$ is the center of the sphere and $r$ is the radius of the sphere.

Given the points $p_i = (x_i, y_i, z_i)$ for $i = 1$ to 3 on the surface of a sphere of radius $r$, the following three equations must be satisfied:

$$(x_1-a)^2 + (y_1-b)^2 + (z_1-c)^2 = r^2, \tag{4}$$
$$(x_2-a)^2 + (y_2-b)^2 + (z_2-c)^2 = r^2, \tag{5}$$
$$(x_3-a)^2 + (y_3-b)^2 + (z_3-c)^2 = r^2. \tag{6}$$

If we subtract eq. 5 from eq. 4 we obtain

$$\begin{aligned}(x_2-x_1)a + (y_2-y_1)b + (z_2-z_1)c + \\ 0.5({x_1}^2 - {x_2}^2 + {y_1}^2 - {y_2}^2 + {z_1}^2 - {z_2}^2) = 0.\end{aligned} \tag{7}$$

If we subtract eq. 6 from eq. 4 we obtain

$$\begin{aligned}(x_3-x_1)a + (y_3-y_1)b + (z_3-z_1)c + \\ 0.5({x_1}^2 - {x_3}^2 + {y_1}^2 - {y_3}^2 + {z_1}^2 - {z_3}^2) = 0.\end{aligned} \tag{8}$$

Using eq. 7 and eq. 8, we can solve for $a$ and $b$ in terms of $c$, namely,

$$\begin{aligned}b = [2(-x_2z_1 + x_3z_1 + x_1z_2 - x_3z_2 - x_1z_3 + x_2z_3)c + {x_1}^2x_2 \\ -x_1{x_2}^2 - {x_1}^2x_3 + {x_2}^2x_3 + x_1{x_3}^2 - x_2{x_3}^2 + x_2{y_1}^2 - x_3{y_1}^2 \\ -x_1{y_2}^2 + x_3{y_2}^2 + x_1{y_3}^2 - x_2{y_3}^2 + x_2{z_1}^2 - x_3{z_1}^2 - x_1{z_2}^2 \\ +x_3{z_2}^2 + x_1{z_3}^2 - x_2{z_3}^2]M\end{aligned} \tag{9}$$

and

$$\begin{aligned}a = [2(y_2z_1 - y_3z_1 - y_1z_2 + y_3z_2 + y_1z_3 - y_2z_3)c + {x_2}^2y_1 \\ -{x_3}^2y_1 - {x_1}^2y_2 + {x_3}^2y_2 - {y_1}^2y_2 + y_1{y_2}^2 + {x_1}^2y_3 - {x_2}^2y_3 \\ +{y_1}^2y_3 - {y_2}^2y_3 - y_1{y_3}^2 + y_2{y_3}^2 - y_2{z_1}^2 + y_3{z_1}^2 + y_1{z_2}^2 \\ -y_3{z_2}^2 - y_1{z_3}^2 + y_2{z_3}^2]M\end{aligned} \tag{10}$$

where $M = 1/[2(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3)]$.

Equations 9 and 10 can be substituted into eq. 4 to leave a quadratic equation in terms of the variable $c$. This equation can be solved for the two values of $c$. These values can be substituted back into eqs. 9 and 10 to obtain two center points for the sphere. Let the center points be $Center_1 = (a_1, b_1, c_1)$ and $Center_2 = (a_2, b_2, c_2)$.

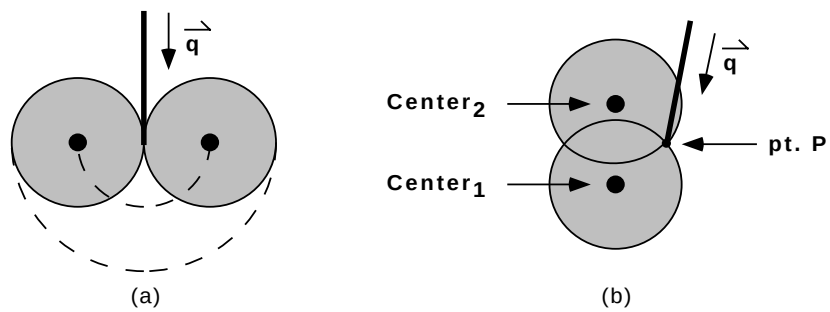


**Fig. 3.** Elimination of one of two sphere centers

If a touch sensor is being used to make contact with the sphere, then the knowledge of the orientation of the sensor relative to the two possible spheres may possibly eliminate one of the two mathematical solutions, hence, leaving the correct solution. Let $\mathbf{q}$ be the vector pointing in the direction of the touch sensor. Let $p$ be the position of the touch to the surface of the sphere. From Figure 3a, it can be seen that the center of the sphere must be located "lower" than point $p$. From Figure 3b, one of the two spheres cannot be possible because the touch sensor must intersect that sphere to reach point $p$. Therefore, we can eliminate the sphere that does not satisfy the following equation:

$$(p - Center_i) \cdot \mathbf{q} < 0. \tag{11}$$

If both sphere centers pass this test for all three points, then another point will be necessary to eliminate one of the sphere centers.

### 4.2 An Example

If we let $r = 2$, $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$, $p_3 = (-1, 0, 0)$, and $\mathbf{q} = (-4, 1, 1)$ for point $p_1$, then eqs. 4, 5, and 6 become

$$(1 - a)^2 + (b)^2 + (c)^2 = 4, \tag{12}$$

$$(a)^2 + (1 - b)^2 + (c)^2 = 4, \tag{13}$$

$$(-1 - a)^2 + (b)^2 + (c)^2 = 4. \tag{14}$$

From eqs. 7 and 8 we find that $a = b = 0$. Plugging these results into eq. 12, we find $c = \pm\sqrt{3}$. Therefore, our possible center points are $Center_1 = (0, 0, +\sqrt{3})$ or $Center_2 = (0, 0, -\sqrt{3})$. Using $\mathbf{q}$, $Center_1$ cannot be possible because

$$\begin{aligned} [P_1 - Center_1] \cdot \mathbf{q} &= [(1, 0, 0) - (0, 0, \sqrt{3})] \cdot (-4, 1, 1) \\ (1, 0, -\sqrt{3}) \cdot (1, 1, -4) &= 1 + 4\sqrt{3} > 0. \end{aligned} \tag{15}$$

Hence, the sphere center must be $Center_2 = (0, 0, -\sqrt{3})$.

# 5. Determination of the Location of a Cylinder Using Five Points

The Location of a cylinder of known radius, in general, can be found in three-dimensional Euclidean space if five points on the surface of the cylinder are known. There are several different ways of representing a cylinder. However, a method used by Schaal (1985) develops algebraic equations using a minimal set of variables. This method has been extended for application here. In the next section, the algebraic equations for finding a cylinder are derived. An example using the derived method is also given.

## 5.1 Equation Formulation

In order to find the location of a cylinder in space using a finite number of points on its surface, a general equation for a cylinder needs to be formulated. We use the equation developed by Schaal (1985) since it describes a cylinder in a form suitable for our subsequent developments. Here we give a derivation of this equation.

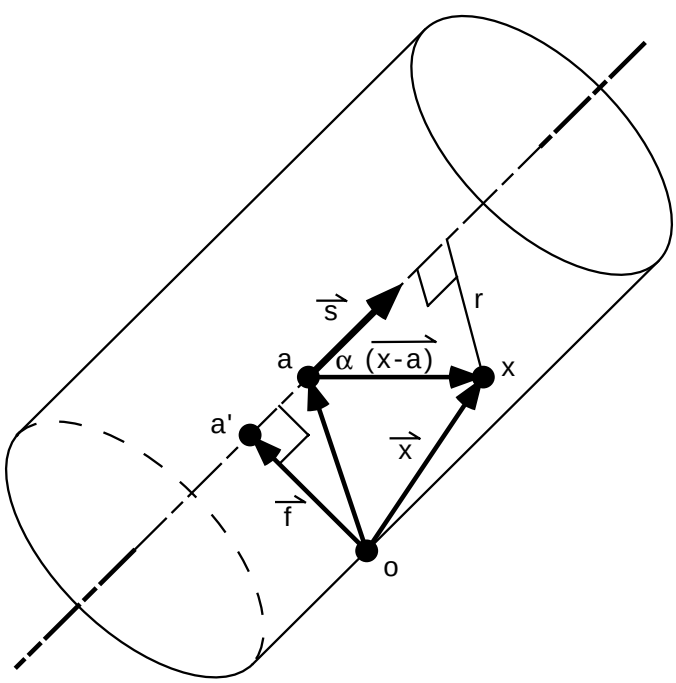


**Fig. 4.** Analytical model of a cylinder

In Figure 4, a cylinder is given with the following properties: $r$ is the radius of the cylinder, $x$ and $o$ are points on the surface of the cylinder, $a$ and $a'$ are points on the axis of the cylinder, **s** is a vector in the direction of the axis of the cylinder, $\alpha$ is the angle between **s** and (x-a), **x** is a vector from point $o$ to point $x$, and **f** is a

vector from point $o$ to the axis of the cylinder where $\mathbf{f}$ is perpendicular to $\mathbf{s}$. From Figure 4 and vector analysis it is obvious that

$$r = |(x-a)| \sin \alpha. \tag{16}$$

Also from vector analysis we know that the cross product between two vectors is equal in magnitude to the product of the magnitudes of the vectors times the sine of the angle between the two vectors. Using this property and eq. 16 we obtain,

$$|(x-a) \times \mathbf{s}| = |(x-a)||\mathbf{s}| \sin \alpha = (|(x-a)| \sin \alpha)|\mathbf{s}| = r|\mathbf{s}|. \tag{17}$$

Letting $||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}$, eq. 17 can be written as,

$$||(x-a) \times \mathbf{s}||^2 = r^2||\mathbf{s}||^2 \quad or \quad ||(x-a) \times \mathbf{s}||^2 - r^2||\mathbf{s}||^2 = 0. \tag{18}$$

Equation 18 is a general equation for a cylinder. We will now proceed to change this equation to only leave the variable $\mathbf{s}$.

The left side of eq. 18, substituting $(x-a) = \mathbf{x} - \mathbf{f}$, can be written as

$$||(x-a) \times \mathbf{s}||^2 = [(\mathbf{x} - \mathbf{f}) \times \mathbf{s}] \cdot [(\mathbf{x} - \mathbf{f}) \times \mathbf{s}]. \tag{19}$$

Using the vector equation

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}), \tag{20}$$

$(\mathbf{x} - \mathbf{f}) \times \mathbf{s}$ can be written as

$$(\mathbf{x} - \mathbf{f}) \times \mathbf{s} = (\mathbf{x} \times \mathbf{s}) - (\mathbf{f} \times \mathbf{s}). \tag{21}$$

Substituting $\boldsymbol{\delta} = \mathbf{f} \times \mathbf{s}$ into the right side of eq. 21, we obtain

$$(\mathbf{x} \times \mathbf{s}) - (\mathbf{f} \times \mathbf{s}) = (\mathbf{x} \times \mathbf{s}) - \boldsymbol{\delta}. \tag{22}$$

Substituting eq. 22 into eq. 19, we obtain

$$||(x-a) \times \mathbf{s}||^2 = [(\mathbf{x} \times \mathbf{s}) - \boldsymbol{\delta}] \cdot [(\mathbf{x} \times \mathbf{s}) - \boldsymbol{\delta}]. \tag{23}$$

Using the vector equation

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}, \tag{24}$$

equation 23 can be written as

$$||(x-a) \times \mathbf{s}||^2 = ||\mathbf{x} \times \mathbf{s}||^2 + ||\boldsymbol{\delta}||^2 - 2[(\mathbf{x} \times \mathbf{s}) \cdot \boldsymbol{\delta}]. \tag{25}$$

We already know that

$$(\mathbf{x} \times \mathbf{s}) \cdot \boldsymbol{\delta} = (\mathbf{x} \times \mathbf{s}) \cdot (\mathbf{f} \times \mathbf{s}) \tag{26}$$

from our definition of $\delta$. Using the vector equation

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}), \tag{27}$$

equation 26 can be written as

$$(\mathbf{x} \times \mathbf{s}) \cdot (\mathbf{f} \times \mathbf{s}) = (\mathbf{x} \cdot \mathbf{f})(\mathbf{s} \cdot \mathbf{s}) - (\mathbf{x} \cdot \mathbf{s})(\mathbf{s} \cdot \mathbf{f}). \tag{28}$$

If we let $\mathbf{s}$ be normal to $\mathbf{f}$ then $(\mathbf{s} \cdot \mathbf{f}) = 0$. This changes eq. 28 to

$$(\mathbf{x} \times \mathbf{s}) \cdot (\mathbf{f} \times \mathbf{s}) = (\mathbf{x} \cdot \mathbf{f})(\mathbf{s} \cdot \mathbf{s}) = (\mathbf{x} \cdot \mathbf{f})||\mathbf{s}||^2. \tag{29}$$

Substituting eq. 29 into eq. 25, we obtain

$$||(x-a)\times\mathbf{s}||^2 = ||\mathbf{x}\times\mathbf{s}||^2 + ||\boldsymbol{\delta}||^2 - 2(\mathbf{x}\cdot\mathbf{f})||\mathbf{s}||^2. \tag{30}$$

Substituting eq. 30 into eq. 18, we obtain

$$||\mathbf{x}\times\mathbf{s}||^2 + ||\boldsymbol{\delta}||^2 - 2(\mathbf{x}\cdot\mathbf{f})||\mathbf{s}||^2 - r^2||\mathbf{s}||^2 = 0. \tag{31}$$

Since we let $\mathbf{f}$ be normal to $\mathbf{s}$, the magnitude of $\mathbf{f}$ must be equal to the radius of the cylinder. Therefore, $|\boldsymbol{\delta}| = |(\mathbf{f}\times\mathbf{s})| = |\mathbf{f}||\mathbf{s}|$. Hence,

$$||\boldsymbol{\delta}||^2 - r^2||\mathbf{s}||^2 = 0. \tag{32}$$

Using eq. 32, eq. 31 becomes

$$||\mathbf{x}\times\mathbf{s}||^2 - 2(\mathbf{x}\cdot\mathbf{f})||\mathbf{s}||^2 = 0. \tag{33}$$

Let $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$ be the points on the surface of the cylinder. Let $x_5$ be the point $o$, and let $\mathbf{p}_i = x_i - x_5$ for $i = 1\ \ to\ \ 4$. With these changes, eq. 33 can be written as

$$(\mathbf{p}_i\cdot\mathbf{f}) - \frac{1}{2||\mathbf{s}||^2}||\mathbf{p}_i\times\mathbf{s}||^2 = 0 \quad for\ i = 1\ \ to\ \ 4. \tag{34}$$

Expanding eq. 34 into a component form using $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we obtain

$$\begin{bmatrix} p_{1_x} & p_{1_y} & p_{1_z} \\ p_{2_x} & p_{2_y} & p_{2_z} \\ p_{3_x} & p_{3_y} & p_{3_z} \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} - \frac{1}{2||\mathbf{s}||^2} \begin{bmatrix} ||\mathbf{p}_1\times\mathbf{s}||^2 \\ ||\mathbf{p}_2\times\mathbf{s}||^2 \\ ||\mathbf{p}_3\times\mathbf{s}||^2 \end{bmatrix} = 0. \tag{35}$$

If we let

$$M = \begin{bmatrix} p_{1_x} & p_{1_y} & p_{1_z} \\ p_{2_x} & p_{2_y} & p_{2_z} \\ p_{3_x} & p_{3_y} & p_{3_z} \end{bmatrix}, \tag{36}$$

then

$$M^{-1} = \frac{1}{det[M]} \begin{bmatrix} (\mathbf{p}_2\times\mathbf{p}_3)_x & (\mathbf{p}_3\times\mathbf{p}_1)_x & (\mathbf{p}_1\times\mathbf{p}_2)_x \\ (\mathbf{p}_2\times\mathbf{p}_3)_y & (\mathbf{p}_3\times\mathbf{p}_1)_y & (\mathbf{p}_1\times\mathbf{p}_2)_y \\ (\mathbf{p}_2\times\mathbf{p}_3)_z & (\mathbf{p}_3\times\mathbf{p}_1)_z & (\mathbf{p}_1\times\mathbf{p}_2)_z \end{bmatrix}. \tag{37}$$

Multiplying both sides of eq. 35 by $M^{-1}$ gives

$$\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \frac{W}{2||\mathbf{s}||^2\ det[M]} \begin{bmatrix} ||\mathbf{p}_1\times\mathbf{s}||^2 \\ ||\mathbf{p}_2\times\mathbf{s}||^2 \\ ||\mathbf{p}_3\times\mathbf{s}||^2 \end{bmatrix} \tag{38}$$

where

$$W = \begin{bmatrix} (\mathbf{p}_2\times\mathbf{p}_3)_x & (\mathbf{p}_3\times\mathbf{p}_1)_x & (\mathbf{p}_1\times\mathbf{p}_2)_x \\ (\mathbf{p}_2\times\mathbf{p}_3)_y & (\mathbf{p}_3\times\mathbf{p}_1)_y & (\mathbf{p}_1\times\mathbf{p}_2)_y \\ (\mathbf{p}_2\times\mathbf{p}_3)_z & (\mathbf{p}_3\times\mathbf{p}_1)_z & (\mathbf{p}_1\times\mathbf{p}_2)_z \end{bmatrix}. \tag{39}$$

Let $\mathbf{n}_1 = \mathbf{p}_2\times\mathbf{p}_3$, $\mathbf{n}_2 = \mathbf{p}_3\times\mathbf{p}_1$, and $\mathbf{n}_3 = \mathbf{p}_1\times\mathbf{p}_2$. Substituting $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ into eq. 38 and multiplying both sides of the equation by $\mathbf{s}$ we obtain

$$[\mathbf{s}] \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \frac{\mathbf{s}}{2||\mathbf{s}||^2\ det[M]} \begin{bmatrix} n_{1_x} & n_{2_x} & n_{3_x} \\ n_{1_y} & n_{2_y} & n_{3_y} \\ n_{1_z} & n_{2_z} & n_{3_z} \end{bmatrix} \begin{bmatrix} ||\mathbf{p}_1\times\mathbf{s}||^2 \\ ||\mathbf{p}_2\times\mathbf{s}||^2 \\ ||\mathbf{p}_3\times\mathbf{s}||^2 \end{bmatrix}. \tag{40}$$

Since $\mathbf{s} \cdot \mathbf{f} = 0$, the two sides of eq. 40 must be equal to zero. The right side of the equation now becomes

$$\begin{bmatrix} s_x & s_y & s_z \end{bmatrix} \begin{bmatrix} n_{1_x} & n_{2_x} & n_{3_x} \\ n_{1_y} & n_{2_y} & n_{3_y} \\ n_{1_z} & n_{2_z} & n_{3_z} \end{bmatrix} \begin{bmatrix} ||\mathbf{p}_1 \times \mathbf{s}||^2 \\ ||\mathbf{p}_2 \times \mathbf{s}||^2 \\ ||\mathbf{p}_3 \times \mathbf{s}||^2 \end{bmatrix} = 0. \tag{41}$$

Equation 41 is the equation Schaal (1985) derived for a cylinder. This equation describes a cylinder in three-dimensional Euclidean space using vector $\mathbf{s}$ and four points on the surface of the cylinder.

Vector $\mathbf{s}$ describes the direction of the center line of the cylinder. It can be denoted as $(s_x, s_y, s_z)$. The actual magnitude of the vector is not important for our case; therefore, we can set one of the vector components equal to one. Let $s_z = 1$. For this case, vector $\mathbf{s}$ must not be parallel to the $x$-$y$ plane; if it is, then $s_x$ or $s_y$ will tend to infinity during a calculation of $\mathbf{s}$. If this happens, then either $s_x$ or $s_y$ should be set to one instead of $s_z$. Note that it is unlikely that a vector will have any directional components equal to zero using an actual robot end-effector frame; hence, any real calculations should work with $s_z = 1$.

Using $\mathbf{s} = (s_x, s_y, 1)$ means that there are two unknowns, $s_x$ and $s_y$, yet we only have one equation, eq. 41. Therefore, another equation is necessary for the calculation of $\mathbf{s}$. If we add another point on the surface, $x_4$ where $\mathbf{p}_4 = x_4 - x_5$, to eq. 35, then we obtain

$$\begin{bmatrix} p_{1_x} & p_{1_y} & p_{1_z} \\ p_{2_x} & p_{2_y} & p_{2_z} \\ p_{3_x} & p_{3_y} & p_{3_z} \\ p_{4_x} & p_{4_y} & p_{4_z} \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} - \frac{1}{2||\mathbf{s}||^2} \begin{bmatrix} ||\mathbf{p}_1 \times \mathbf{s}||^2 \\ ||\mathbf{p}_2 \times \mathbf{s}||^2 \\ ||\mathbf{p}_3 \times \mathbf{s}||^2 \\ ||\mathbf{p}_4 \times \mathbf{s}||^2 \end{bmatrix} = 0. \tag{42}$$

From linear algebra (Roberts, 1985), Eq. 42 is only valid if

$$\det \begin{bmatrix} p_{1_x} & p_{1_y} & p_{1_z} & ||\mathbf{p}_1 \times \mathbf{s}||^2 \\ p_{2_x} & p_{2_y} & p_{2_z} & ||\mathbf{p}_2 \times \mathbf{s}||^2 \\ p_{3_x} & p_{3_y} & p_{3_z} & ||\mathbf{p}_3 \times \mathbf{s}||^2 \\ p_{4_x} & p_{4_y} & p_{4_z} & ||\mathbf{p}_4 \times \mathbf{s}||^2 \end{bmatrix} = 0. \tag{43}$$

Equation 43 is a second-degree equation. Using eqs. 41 and 43, vector $\mathbf{s}$ is, in general, solvable using $s_z = 1$. The two equations can be written in the general form

$$\begin{aligned} a{s_x}^3 + b{s_y}^3 + c{s_x}^2 s_y + d s_x {s_y}^2 + e{s_x}^2 + f{s_y}^2 \\ + g s_x s_y + h s_x + i s_y + j = 0 \end{aligned} \tag{44}$$

and

$$k{s_x}^2 + l{s_y}^2 + m s_x s_y + n s_x + p s_y + q = 0. \tag{45}$$

Equations 44 and 45 can be combined to form one equation of degree six in only one variable using Sylvester's Method (Boehm and Prautsch, 1994). This equation can be solved numerically to obtain six possible solutions for one of the variables, $s_x$ or $s_y$. Using eq. 45, the other variable can be found. Therefore, we will have six possible solutions for the cylinder axis direction $\mathbf{s}$. We still need to find a point on the center line of the cylinder to define the location of the cylinder in space. We will do this by using three of the points on the surface to create a circle on a plane perpendicular to the cylinder and find the center of this circle, as shown in Figure 5.

When we find the center of the circle, we can also calculate the radius of the circle, which is also the radius of the cylinder. Since we know the radius of the

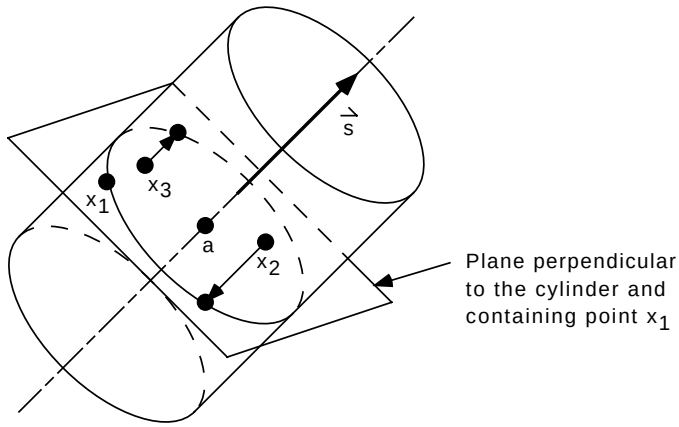


**Fig. 5.** Finding a point on the center line of a cylinder

cylinder already, we can eliminate the cases where the calculated radius does not match the actual radius. This should leave only the correct cylinder result. In the next section, an example is given to illustrate the cylinder finding process.

### 5.2 An Example

Let $x_1$ = (2.927025, 2.000000, 1.625000), $x_2$ = (1.775000, 2.800000, 1.443785), $x_3$ = (1.372975, 2.600000, 3.116620), $x_4$ = (2.075000, 3.40000, 2.185405), and $x_5$ = (3.527025, 3.200000, 3.108240) be five points on the surface of a cylinder of radius $r$ = 1.000000. These points correspond to a cylinder of radius one and a center line (defined using a vector and point) of vector $\mathbf{s}$ = (0.300000, 0.600000, 0.741620) and point $O$ = (2, 2, 2). Since the answer is known, the results of the example can be verified. Using equation $\mathbf{p}_i = x_i - x_5$ for $i$ = 1 to 4, we find $\mathbf{p}_1$ = (-0.600000, -1.200000, -1.483240), $\mathbf{p}_2$ = (-1.752025, -0.400000, - 1.664455), $\mathbf{p}_3$ = (-2.154050, -0.600000, 0.008380), and $\mathbf{p}_4$ = (-1.452025, 0.200000, -0.922835).

Using the values for $\mathbf{p}_i$, eq. 41 written in the form of eq. 44 becomes

$$\begin{aligned} -1.785361 - 5.257678s_x + 3.124352s_x^2 - 0.504446s_x^3 \\ +0.489010s_y + 16.119020s_x s_y + 0.854990s_x^2 s_y - 3.701228s_y^2 \\ +4.371015s_x s_y^2 - 2.048000s_y^3 = 0. \end{aligned} \tag{46}$$

Using the values for $\mathbf{p}_i$, eq. 43 written in the form of eq. 45 becomes

$$\begin{aligned} 0.724860 - 9.049719s_x + 0.875140s_x^2 + 0.316760s_y \\ -5.049719s_x s_y + 6.400000s_y^2 = 0. \end{aligned} \tag{47}$$

Using an equation-solver, eqs. 46 and 47 are solved for $\mathbf{s}$. Six solutions exist for this case, two of which are real, namely, $\mathbf{s}$ = (-257.042261,-71.59786,1.00000) and $\mathbf{s}$ = (0.40452,0.80904,1.00000).

Using the values of $\mathbf{s}$, we calculated the values for the radius of the cylinder and a point on the axis of the cylinder. Table 1 shows the results. We know the radius for this particular case is exactly one. Hence, the case that has a radius that is very close to one should be the correct cylinder. The case with $\mathbf{s}$ =

**Table 1.** List of possible cylinder locations

| **s** | Radius of cyl. | Point on the axis |
|---|---|---|
| $(-257, -71.6, 1)$ | $r = 0.9165$ | $(2.751, 2.640, 2.257)$ |
| $(0.405, 0.809, 1)$ | $r = 1.00000$ | $(2, 2, 2)$ |

(0.404520,0.809040,1.000000) has a mathematically determined radius of 1.000000. Therefore, it is the correct cylinder. This matches the expected result (note: **s** needs to be normalized to match the expected result).

If the actual radius of the cylinder was not known ahead of time, then an additional point on the cylinder would be needed to eliminate the incorrect solutions. However, this would not happen when working with reference fixtures because the user designs the fixture.

## 6. Fixture Analysis Example

We now have the computational tools to find a plane, a sphere, and a cylinder in space. To emphasize the use of these methods, we will give an example of a fixture with feature surfaces consisting of a sphere and a cylinder (a cylinder-sphere fixture). Figure 6 shows a possible design of a cylinder-sphere fixture. The limited touch area on the cylinder is needed to determine the direction of **Y**. Vector **Y** must be parallel to the axis of the cylinder; however, this leaves two possible directions. By limiting the touch sensitive surface of the cylinder to be above or below the sphere, we can always pick the correct direction.

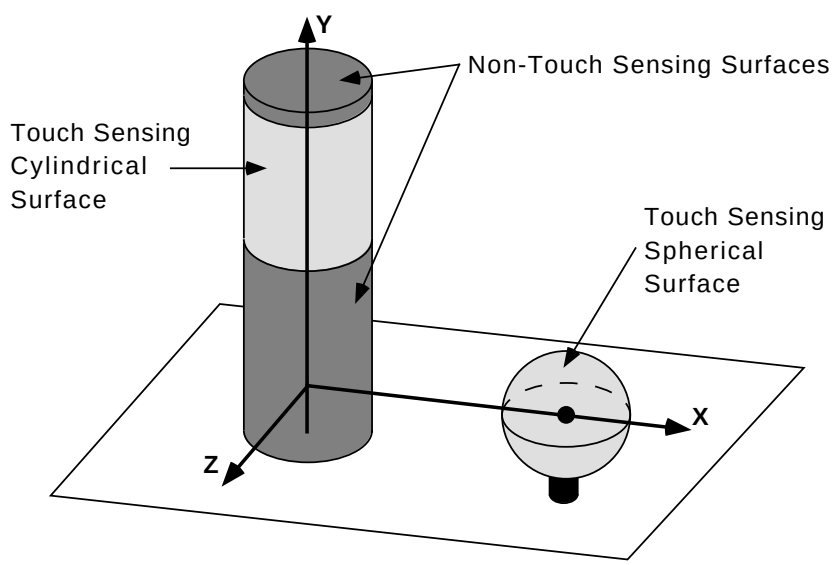


**Fig. 6.** A cylinder-sphere fixture and its frame

The creation of a reference frame from the geometric elements is quite simple as can be seen in Figure 6. Vector **Y** points along the axis of the cylinder, vector **X** is parallel to the shortest line between the axis of the cylinder and the center of the sphere, and vector **Z** is the cross product $\mathbf{X} \times \mathbf{Y}$. The origin of the frame is the intersection between the axis of the cylinder and the line created by finding the shortest distance between the center of the sphere and the axis of the cylinder.

Based on the frame construction just described, the information needed to create the frame is the location of the axis of the cylinder and the center of the sphere. Therefore, given five point-contact locations on the cylinder and three point-contact locations on the sphere, we should be able to create a reference frame using the methods described earlier. For this case, if the shortest distance between the sphere and the cylinder is known, then one less point on the sphere is necessary. In the area of fixture design, the fixture is designed; therefore, the distance between all of its geometric elements are known. This is shown in the example calculations.

For this example we will use the cylinder values from the cylinder analysis in the previous section. Hence, we know that $\mathbf{s} = (0.300000,\ 0.600000,\ 0.741620)$ and $O = (2,\ 2,\ 2)$. Let $x_6 = (5.456074,\ 4.600000,\ 2.543645)$ and $x_7 = (5.981074,\ 4.400000,\ 3.841479)$. The points $x_6$ and $x_7$ are two points on the surface of the sphere. The analysis model for this cylinder-sphere example is shown in Figure 7. Let the shortest distance $d$ between the sphere and the axis of the cylinder be 3.000000. All of these values were chosen to correspond to a sphere of radius one with a center located at (5.681074, 3.800000, 3.099860). In our example we will let the center of the sphere be $(c_x, c_y, c_z)$, and, using the given values, try to find the correct center value.

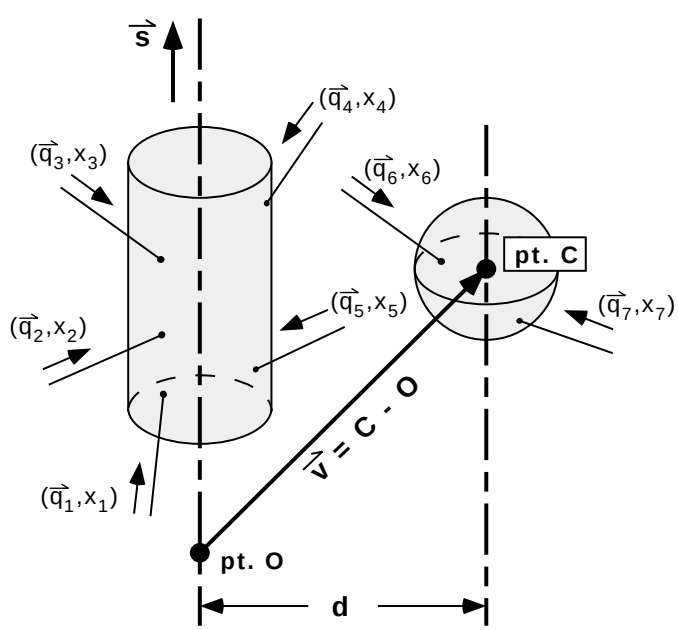


**Fig. 7.** Analysis of a cylinder-sphere fixture

We know that the two points on the sphere must satisfy the general equation for a sphere of unit radius. Therefore, the following equations must hold:

$$(5.456074 - c_x)^2 + (4.600000 - c_y)^2 + (2.543645 - c_z)^2 = 1. \tag{48}$$

$$(5.981074 - c_x)^2 + (4.400000 - c_y)^2 + (3.841479 - c_z)^2 = 1. \tag{49}$$

Also the shortest distance between the axis of the cylinder and the center of the sphere must be 3.00000. From Figure 7 and vector analysis, we know

$$d = \frac{|\mathbf{s} \times \mathbf{v}|}{|\mathbf{s}|}. \tag{50}$$

Vector $\mathbf{s}$ is given as a unit vector for our calculations. Therefore, the denominator for eq. 50 is one. Equation 50 can be rewritten as

$$d^2 = (\mathbf{s} \times \mathbf{v}) \cdot (\mathbf{s} \times \mathbf{v}). \tag{51}$$

Equations 48, 49, and 51 can be solved to find the possible values of the sphere center. However, if eq. 48 is subtracted from eq. 49, then a linear equation is produced. If this resultant equation is substituted for one of the quadratic sphere equations, then there will be only four solutions instead of eight. Therefore, this is a good computational simplification.

Solving these equations we get four solutions, two of which are real. They are (5.600545, 5.180286, 3.345141) and (5.681074, 3.80000, 3.09986). Obviously, the second solution is the one we are looking for. If we have the probe orientation vectors for each contact $\mathbf{q}_i$, then we may be able to eliminate the incorrect solution as we did in the previous sphere example. With the correct sphere center and the axis of the cylinder known, the complete reference frame can be created.

## 7. Conclusion

In this paper, we presented computational methods encountered in kinematic design of tactile sensing fixtures. The methods given make it possible to localize spheres, planes, and cylinders in space using relatively simple algebraic-geometric principles. A detailed example of a cylinder-sphere fixture was given to illustrate these ideas. The results, in addition to their applications in fixture design and analysis, can be applied in other areas of kinematics, including in three-dimensional Burmester theory.

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